

Problem 1. Solve the following initial value problem: $xy' - \frac{y^2}{x^2} = 0$, $y(1) = 1$.

This is a separable d.e.

$$xy' - \frac{y^2}{x^2} = 0 \Rightarrow x \frac{dy}{dx} = \frac{y^2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{y^2}{x^3} \Rightarrow dy = \frac{y^2}{x^3} dx \Rightarrow \frac{dy}{y^2} = \frac{dx}{x^3}$$

$$\Rightarrow \int y^{-2} dy = \int x^{-3} dx \Rightarrow \frac{y^{-1}}{-1} = \frac{x^{-2}}{-2} + c \Rightarrow \frac{1}{y} = \frac{1}{2x^2} - c$$

$$y(1) = 1 \Rightarrow \frac{1}{1} = \frac{1}{2(1)^2} - c \Rightarrow c = -\frac{1}{2} \boxed{1 \text{ pt.}} \Rightarrow \frac{1}{y} = \frac{1}{2x^2} + \frac{1}{2} = \frac{1+x^2}{2x^2} \Rightarrow \boxed{y = \frac{2x^2}{x^2+1}}$$

Problem 2. Solve the following initial value problem: $xyy' + y^2 - x^2 = 0$, $y(2) = 1$.

$xyy' + y^2 - x^2 = 0 \Rightarrow y' = \frac{x^2 - y^2}{xy}$. Since y' equals a rational function in which each term has the same degree (2), the d.e. is homogeneous.

We introduce the new variable $v = y/x$. In the d.e. we replace y' by $v + x \frac{dv}{dx}$ and we replace y by xv :

$$y' = \frac{x^2 - y^2}{xy} \Rightarrow v + x \frac{dv}{dx} = \frac{x^2 - (xv)^2}{x(xv)} = \frac{x^2(1-v^2)}{x^2v} = \frac{1-v^2}{v} \Rightarrow x \frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}$$

$$\Rightarrow \int \underbrace{\frac{v}{1-2v^2}}_{\text{Let } u=1-2v^2} dv = \int \frac{dx}{x} \Rightarrow -\frac{1}{4} \ln(1-2v^2) = \ln(x) + c \Rightarrow e^{-\frac{1}{4} \ln(1-2v^2)} = e^{\ln(x)+c} \Rightarrow$$

$$(1-2v^2)^{-1/4} = e^{\ln(x)} \underbrace{e^c}_{c_1} = c_1 x \Rightarrow 1-2v^2 = \underbrace{c_1^{-4}}_{c_2} x^{-4} = \frac{c_2}{x^4} \Rightarrow 1-2(y/x)^2 = \frac{c_2}{x^4}$$

$$y(2) = 1 \Rightarrow 1-2(1/2)^2 = \frac{c_2}{2^4} \Rightarrow \frac{1}{2} = \frac{c_2}{16} \Rightarrow c_2 = 8$$

Therefore, $1-2(y/x)^2 = \frac{8}{x^4} \Rightarrow x^4 - 2x^2y^2 = 8 \Rightarrow \boxed{y = \frac{\sqrt{x^4-8}}{\sqrt{2}x}}$.

Problem 3. Solve the following initial value problem: $y' - \frac{4y}{x} = x^4 \cos(x)$, $y(\pi) = 0$.

This is a linear d.e. because y and y' appear to the first power and are multiplied by constants or functions of x .

The d.e. is already in standard form.

Find the integrating factor: $\rho(x) = e^{\int P(x) dx} = e^{\int -4/x dx} = e^{-4 \ln(x)} = x^{-4}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$$x^{-4} \left[\frac{dy}{dx} - \left(\frac{4}{x} \right) y \right] = x^{-4} (x^4 \cos(x)) = \cos(x) \Rightarrow x^{-4} \frac{dy}{dx} - 4x^{-5}y = \cos(x).$$

Use the Product Rule backwards to rewrite the d.e. as $\frac{d}{dx} [x^{-4}y] = \cos(x)$.

Integrating both sides, we obtain $x^{-4}y = \int \cos(x) dx = \sin(x) + c$.

$$y(\pi) = 0 \Rightarrow \pi^{-4}(0) = \sin(\pi) + c = c \Rightarrow c = 0$$

Therefore, $x^{-4}y = \sin(x)$, so $y = x^4 \sin(x)$.

Problem 4. Solve the following initial value problem: $2xyy' + y^2 - 4x^3 = 0$, $y(1) = 2$.

The given d.e. is exact: $\underbrace{2xy y'}_N + \underbrace{y^2 - 4x^3}_M = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [y^2 - 4x^3] = 2y \text{ and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [2xy] = 2y. \text{ The d.e. is exact because } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Therefore, the solution of the d.e. is $f(x, y) = c$ where f satisfies the conditions $\frac{\partial f}{\partial x} = M = y^2 - 4x^3$ and $\frac{\partial f}{\partial y} = N = 2xy$.

$$\frac{\partial f}{\partial x} = y^2 - 4x^3 \Rightarrow f = \int (y^2 - 4x^3) dx = xy^2 - x^4 + g(y) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [xy^2 - x^4 + g(y)] = 2xy + g'(y).$$

But $\frac{\partial f}{\partial y} = N = 2xy$. Therefore, $2xy + g'(y) = 2xy \Rightarrow g'(y) = 0 \Rightarrow g(y) = \int 0 dy = 0$ so $f = xy^2 - x^4$. The solution of the d.e. is $xy^2 - x^4 = c$

$$y(1) = 2 \Rightarrow 1(2)^2 - 1^4 = c \Rightarrow c = 3 \Rightarrow xy^2 - x^4 = 3 \Rightarrow y^2 = \frac{x^4 + 3}{x} \Rightarrow y = \sqrt{\frac{x^4 + 3}{x}}$$

Problem 5. Let P denote the population of a colony of tribbles. Suppose that β (the number of births per week per tribble) is proportional to \sqrt{P} and that δ (the number of deaths per week per tribble) equals 0. Suppose the initial population is 4 and the population after 1 week is 9. What is the population after 2 weeks?

Recall that the de modeling population problems is $\frac{dP}{dt} = \beta P - \delta P$.

Let t denote time (in weeks). $\beta = k\sqrt{P}$ and $\delta = 0$ so $\frac{dP}{dt} = (k\sqrt{P}) P - 0 \cdot P = kP^{3/2}$

This d.e. is separable: $\frac{dP}{dt} = kP^{3/2} \Rightarrow \frac{dP}{P^{3/2}} = k dt \Rightarrow \int P^{-3/2} dP = \int k dt \Rightarrow \frac{P^{-1/2}}{-1/2} = kt + c$

$$\Rightarrow -\frac{2}{\sqrt{P}} = kt + c$$

$$P(0) = 4 \Rightarrow -\frac{2}{\sqrt{4}} = k(0) + c \Rightarrow c = -1 \Rightarrow -\frac{2}{\sqrt{P}} = kt - 1 \Rightarrow \frac{2}{\sqrt{P}} = 1 - kt$$

$$P(1) = 9 \Rightarrow \frac{2}{\sqrt{9}} = 1 - k(1) \Rightarrow \frac{2}{3} = 1 - k \Rightarrow k = \frac{1}{3} \Rightarrow \frac{2}{\sqrt{P}} = 1 - \frac{t}{3}$$

Therefore, $\frac{2}{\sqrt{P(2)}} = 1 - \frac{2}{3} = \frac{1}{3} \Rightarrow \frac{4}{P(2)} = \frac{1}{9} \Rightarrow P(2) = 36$ tribbles

Problem 6. Find the general solution to each of the following linear homogeneous differential equations:

a. $y''' + 2y'' + 2y' = 0$

The characteristic equation is $r^3 + 2r^2 + 2r = 0 \Rightarrow r(r^2 + 2r + 2) = 0$

$$r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Therefore, the roots of the characteristic equation are $r = 0$ and $r = -1 \pm i$

so the solution of the d.e. is

$$y = c_1 e^{0x} + c_2 e^{-1x} \cos(1x) + c_3 e^{-1x} \sin(1x), \text{ or } \boxed{y = c_1 + c_2 e^{-x} \cos(x) + c_3 e^{-x} \sin(x)}.$$

b. $y^{(4)} - 9y'' = 0$

The characteristic equation is $r^4 - 9r^2 = 0 \Rightarrow r^2(r^2 - 9) = 0 \Rightarrow r^2(r + 3)(r - 3) = 0 \Rightarrow$

$r = 0$ (double root), $r = -3$, or $r = 3$

Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{-3x} + c_4 e^{3x}$, or

$$\boxed{y = c_1 + c_2 x + c_3 e^{-3x} + c_4 e^{3x}}.$$

Problem 7. Consider a forced, damped mass-spring system with mass 1 kg, damping coefficient 2 Ns/m, spring constant 4 N/m, and an external force $F_{\text{ext}}(t) = 8 \cos(2t)$ N. Find the steady-state periodic solution $x_{\text{sp}}(t)$.

The d.e. modeling this system is $m x'' + c x' + k x = F_e(t)$, or $x'' + 2x' + 4x = 8 \cos(2t)$.

The steady-state periodic solution $x_{\text{sp}}(t)$ is the particular solution $x_p(t)$.

The nonhomogeneous term in the de is $8 \cos(2t)$, a cosine function. We should therefore guess that x_p is a combination of a cosine function and a sine function with the same coefficient of t : $x_p = A \cos(2t) + B \sin(2t)$. This guess does not duplicate any term x_c because x_c is a transient term containing decaying exponential functions, so there is no need to modify this guess.

$$x = A \cos(2t) + B \sin(2t) \Rightarrow x' = -2A \sin(2t) + 2B \cos(2t) \Rightarrow x'' = -4A \cos(2t) - 4B \sin(2t)$$

$$\text{Therefore, the left side of the d.e. is } x'' + 2x' + 4x = -4A \cos(2t) - 4B \sin(2t) + 2[-2A \sin(2t) + 2B \cos(2t)] + 4[A \cos(2t) + B \sin(2t)] = 4B \cos(2t) - 4A \sin(2t).$$

We want this to equal the nonhomogeneous term $8 \cos(2t)$:

$$4B \cos(2t) - 4A \sin(2t) = 8 \cos(2t) \Rightarrow 4B = 8, -4A = 0 \Rightarrow A = 0, B = 2. \text{ Thus, } \boxed{x_{\text{sp}} = x_p = 2 \sin(2t)}.$$

Problem 8. Consider an *RLC* circuit with inductance $L = 1$ henry, resistance $R = 5\Omega$, capacitance $C = 0.25$ farads, and applied voltage $E(t) = 20 \cos(2t)$ volts. Suppose the initial charge on the capacitor $Q(0) = 3$ coul and the initial current in the circuit $Q'(0) = 4$ amps. Find the current in the circuit $I(t)$.

The d.e. modeling this system is $LQ'' + RQ' + \frac{Q}{C} = E(t)$, or $Q'' + 5Q' + 4Q = 20 \cos(2t)$.

Step 1. Find Q_c by solving the d.e. $Q'' + 5Q' + 4Q = 0$.

Characteristic equation: $r^2 + 5r + 4 = 0 \Rightarrow (r + 4)(r + 1) = 0 \Rightarrow r = -4$ or $r = -1$

Therefore, $Q_c = c_1 e^{-4t} + c_2 e^{-t}$.

Step 2. Find Q_p . You can use either of the following methods.

Method 1: Undetermined Coefficients. The nonhomogeneous term in the de is $20 \cos(2t)$, a cosine function. We should therefore guess that Q_p is a combination of a cosine function and a sine

function with the same coefficient of t : $Q_p = A \cos(2t) + B \sin(2t)$. This guess does not duplicate any term Q_c , so there is no need to modify this guess.

$$Q = A \cos(2t) + B \sin(2t) \Rightarrow Q' = -2A \sin(2t) + 2B \cos(2t) \Rightarrow Q'' = -4A \cos(2t) - 4B \sin(2t)$$

$$\text{Therefore, the left side of the d.e. is } Q'' + 5Q' + 4Q = -4A \cos(2t) - 4B \sin(2t) + 5[-2A \sin(2t) + 2B \cos(2t)] + 4[A \cos(2t) + B \sin(2t)] = 10B \cos(2t) - 10A \sin(2t).$$

We want this to equal the nonhomogeneous term $20 \cos(2t)$:

$$10B \cos(2t) - 10A \sin(2t) = 20 \cos(2t) \Rightarrow 10B = 20, -10A = 0 \Rightarrow A = 0, B = 2. \text{ Thus, } Q_p = 2 \sin(2t).$$

Method 2: Variation of Parameters. From Q_c we obtain two independent solutions of the homogeneous d.e: $Q_1 = e^{-4t}$ and $Q_2 = e^{-t}$. The Wronskian is given by

$$W(x) = \begin{vmatrix} Q_1 & Q_2 \\ Q_1' & Q_2' \end{vmatrix} = \begin{vmatrix} e^{-4t} & e^{-t} \\ -4e^{-4t} & -e^{-t} \end{vmatrix} = e^{-4t}(-e^{-t}) - (-4e^{-4t})(e^{-t}) = 3e^{-5t}$$

$$u_1 = \int \frac{-Q_2 F_{\text{ext}}(t)}{W(t)} dt = - \int \frac{e^{-t} (20 \cos(2t))}{3e^{-5t}} dt = -\frac{20}{3} \int e^{5t} \cos(2t) dt =$$

$$-\frac{20}{3} \left[\frac{e^{4t}}{4^2 + 2^2} (4 \cos(2t) + 2 \sin(2t)) \right] = e^{4t} \left[-\frac{4}{3} \cos(2t) - \frac{2}{3} \sin(2t) \right] \text{ using formula 50 from the integral table.}$$

$$u_2 = \int \frac{Q_1 F_{\text{ext}}(t)}{W(t)} dt = \int \frac{e^{-4t} (20 \cos(2t))}{3e^{-5t}} dt = \frac{20}{3} \int e^t \cos(2t) dt =$$

$$\frac{20}{3} \left[\frac{e^t}{1^2 + 2^2} (1 \cos(2t) + 2 \sin(2t)) \right] = e^t \left[\frac{4}{3} \cos(2t) + \frac{8}{3} \sin(2t) \right] \text{ using formula 50 from the integral table.}$$

$$\text{Therefore, } Q_p = u_1 Q_1 + u_2 Q_2 = e^{4t} \left[-\frac{4}{3} \cos(2t) - \frac{2}{3} \sin(2t) \right] (e^{-4t}) + e^t \left[\frac{4}{3} \cos(2t) + \frac{8}{3} \sin(2t) \right] (e^{-t}) = 2 \sin(2t)$$

Step 3. $Q = Q_c + Q_p$, so $Q = c_1 e^{-4t} + c_2 e^{-t} + 2 \sin(2t)$

Step 4. Use the initial conditions to determine the values of c_1 and c_2 .

$$Q = c_1 e^{-4t} + c_2 e^{-t} + 2 \sin(2t) \Rightarrow Q' = -4c_1 e^{-4t} - c_2 e^{-t} + 4 \cos(2t)$$

$$Q(0) = 1 \Rightarrow c_1 e^0 + c_2 e^0 + 2 \sin(0) = 1 \Rightarrow c_1 + c_2 = 1.$$

$$Q'(0) = 0 \Rightarrow -4c_1 e^0 - c_2 e^0 + 4 \cos(0) = 0 \Rightarrow -4c_1 - c_2 = -4.$$

$$c_1 + c_2 = 1, -4c_1 - c_2 = -4 \Rightarrow c_1 = 1, c_2 = 0$$

$$\text{Therefore, } Q = e^{-4t} + 2 \sin(2t) \text{ so } \boxed{I = Q' = -4e^{-4t} + 4 \cos(2t)}$$

Problem 9. Use the Laplace Transform to solve the following IVP: $x'' + 5x' + 6x = 4e^{-t}$, $x(0) = 1$, $x'(0) = 0$.

Solutions not using the Laplace transform method will not receive any credit. x is a function of t .

$$x'' \text{ means } \frac{d^2 x}{dt^2}.$$

$$x'' + 5x' + 6x = 4e^{-t} \Rightarrow \mathcal{L}\{x'' + 5x' + 6x\} = \mathcal{L}\{4e^{-t}\} \Rightarrow \mathcal{L}\{x''\} + 5\mathcal{L}\{x'\} + 6\mathcal{L}\{x\} = 4\mathcal{L}\{e^{-t}\} = \frac{4}{s+1}$$

$$\Rightarrow [s^2 \mathcal{L}\{x\} - sx(0) - x'(0)] + 5[s\mathcal{L}\{x\} - x(0)] + 6\mathcal{L}\{x\} = \frac{4}{s+1}$$

$$[s^2 \mathcal{L}\{x\} - s \cdot 1 - 0] + 5[s\mathcal{L}\{x\} - 1] + 6\mathcal{L}\{x\} = \frac{4}{s+1} \Rightarrow$$

$$s^2 \mathcal{L}\{x\} - s + 5s \mathcal{L}\{x\} - 5 + 6\mathcal{L}\{x\} = \frac{4}{s+1} \Rightarrow (s^2 + 5s + 6) \mathcal{L}\{x\} = \frac{4}{s+1} + s + 5 = \frac{s^2 + 6s + 9}{s+1} \Rightarrow$$

$$\mathcal{L}\{x\} = \frac{s^2 + 6s + 9}{(s+1)(s^2 + 5s + 6)} \Rightarrow x = \mathcal{L}^{-1} \left\{ \frac{s^2 + 6s + 9}{(s+1)(s^2 + 5s + 6)} \right\}.$$

Use a partial fraction decomposition:

$$\frac{s^2 + 6s + 9}{(s+1)(s^2 + 5s + 6)} = \frac{(s+3)^2}{(s+1)(s+2)(s+3)} = \frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$(s+1)(s+2) \left[\frac{s+3}{(s+1)(s+2)} \right] = (s+1)(s+2) \left[\frac{A}{s+1} + \frac{B}{s+2} \right] \Rightarrow$$

$$s+3 = A(s+2) + B(s+1) = (A+B)s + (2A+B)$$

$$\Rightarrow A+B=1, 2A+B=3 \Rightarrow A=2, B=-1.$$

$$\text{Therefore, } x = \mathcal{L}^{-1} \left\{ \frac{2}{s+1} + \frac{-1}{s+2} \right\} = 2\mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} \Rightarrow \boxed{x = 2e^{-t} - e^{-2t}}$$