92.236 Engineering Differential Equations Project Fall 2016

DUE DATES: FRIDAY, OCTOBER 21.
OPTIONAL REVISION DUE MONDAY, NOVEMBER 21.

No extensions will be given except in case of illness or other emergency.

General Guidelines

1. You may work in groups of up to four. Turn in just one paper for the whole group. Each member of the group will receive the same grade. **If you would like to work in a group but cannot find a group to join, please notify me no later than Monday, October 3.**

2. If you choose to turn in a revised version of your report, 90% of your grade will be based on the final version and 10% on the first version.

3. This is a major assignment, and you should plan to do some work on it every day, starting today. **DO NOT WAIT UNTIL THE LAST MINUTE.**

4. The paper you turn in must be your group's work. I reserve the right to ask any or all of you for a verbal explanation of your solution.

5. I strongly encourage you to see me regularly as you work on the project to discuss your progress. If you receive any assistance from anyone other than me, of if you use our textbook or any other outside source, you must cite the source in your paper. Remember guideline number 4.

What Should You Hand In?

After you have completely solved all parts of the assigned problem, you should write a report explaining your solution. The report should contain a mixture of text, equations, and graphs, and possibly tables and diagrams. Your text should be grammatically correct, and you should use proper punctuation and spelling. If you do not type the paper, please write legibly. Your paper should contain an introduction explaining the problem and should clearly explain each step of your solution. **Assume the reader knows something about differential equations but has not read the project description below.** Please note that you will be graded on the quality of your presentation as well as on the mathematical content. You may find the following checklist helpful. It was adapted from a checklist developed by Dr. Annalisa Crannell of Franklin and Marshall College. Does this paper

1. clearly restate the problem to be solved?

2. clearly label diagrams, tables, and graphs?

3. define all variables used?

4. provide a paragraph explaining how the problem will be approached?

5. explain how each formula is derived or give a reference indicating where it can be found?

6. give acknowledgment where it is due?

In this paper,

7. are the spelling, grammar, and punctuation correct?

8. is the mathematics correct?

9. did you answer all the questions that were asked?

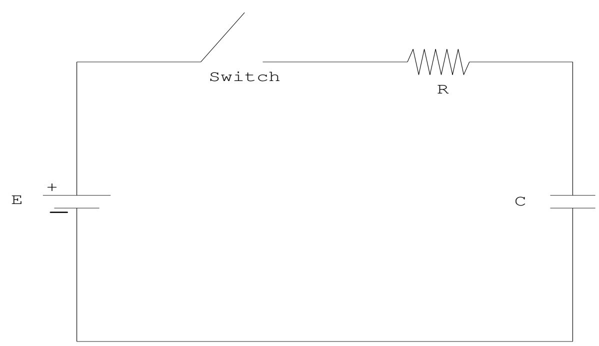
Grade Sheet

1. Derivation of IVP (1) (10 points)	______
2a. Derivation of IVP (2) (10 points)	______
2b. Proof that $v_c = q$ (5 points)	______
3a. Solution of IVP (2) for step function input (10 points)	______
3b. Limiting value of q (5 points)	______
3c. Time for q to reach 95% of limiting value (5 points)	______
3d. Value of q after 1 time constant (5 points)	______
4a. Solution of IVP (2) for sinusoidal input (15 points)	______
4b. Analysis of response to sinusoidal input (5 points)	______
4c. Alternate form of steady-state response (10 points)	______
4d. Plot of amplitude of response vs. input frequency (10 points)	______
5. Report (introduction, clarity and completeness of presentation, grammar). (10 points)	______
Total.	______

Background Information

The purpose of this project is to develop and analyze a mathematical model of an RC circuit. (See the diagram below.) The circuit consists of a voltage source, a resistor of resistance R, and a capacitor of capacitance C. At some instant, which we take to be time 0, the switch is closed, completing the circuit.

We can consider this circuit to be a system to which the input is the applied voltage E and from which the output is the voltage measured across the capacitor, V_C. You will study the response of this system to a constant applied voltage and to a sinusoidal applied voltage.



In the remainder of this project description, the following notation will be used.

T	time (seconds)
Q	charge on the capacitor at time T (coulombs)
I	current in the circuit at time T (amperes)
E	applied voltage (volts)
E_0	maximum value of $\|E\|$ (volts)
V_C	voltage across the capacitor (volts)
V_R	voltage across the resistor (volts)
C	capacitance of the capacitor (farads)
R	resistance of the resistor (ohms)
$t = T/(RC)$	dimensionless time
$q = Q/(E_0 C)$	dimensionless charge on the capacitor
$\epsilon = E/E_0$	dimensionless applied voltage
$v_c = V_C/E_0$	dimensionless voltage across the capacitor

Problem Statement

1. **Formulation of the Model.** Use the following information to formulate an initial value problem (differential equation and initial condition) modeling this circuit. Take T to be the independent variable and Q to be the dependent variable. Do not solve the initial value problem at this stage, just set it up.

   - Kirchoff's Second Law says that the sum of the voltage drops across the resistor and capacitor equals the applied voltage E.
   - The voltage drop across the resistor equals RI.
   - The voltage drop across the capacitor equals Q/C.
   - $I = dQ/dT$.
   - The charge on the capacitor is initially 0.

2. **Nondimensionalization.** A useful technique for analyzing mathematical models is to introduce dimensionless variables. This technique has two advantages: it usually reduces the number of parameters, and it helps makes clear which combination(s) of parameters determine the behavior of the solution. Let $t = T/RC$, let $q = Q/E_0C$, let $\epsilon = E/E_0$, and let $v_c = V_C/E_0$,where E_0 denotes the maximum value of $|E|$ (assumed to be nonzero). Since RC has units of time, t has no units; it is a pure number. Similarly, q, ϵ, and v_c are dimensionless quantities. A value of $t = 1$ corresponds to a time T of one "time constant" RC.

   a. Show that the initial value problem you derived in part 1 can be rewritten in terms of dimensionless variables as follows:

   $$\frac{dq}{dt} + q = \epsilon, \quad q(0) = 0. \tag{1}$$

   b. Show that $v_c = q$.(This means we can consider q to be the response of the system.)

3. **Response to step function input.**

   a. Solve the IVP (1) for $\epsilon = 1$ (which corresponds to a constant input voltage $E = E_0$.)

   b. Show that q approaches a constant value as $t \to \infty$.

   c. How long does it take q to reach 95% of its limiting value?

   d. What fraction of its limiting value does q reach after one time constant ($t = 1$)?

4. **Response to sinusoidal input.**

   a. Solve the IVP (1) for $\epsilon = \cos(\omega t)$ (which corresponds to an input voltage $E = E_0 \cos(\omega T/RC)$).

   b. Show that the response q from part a contains a transient term q_{tr} that approaches 0 as $t \to \infty$ and a steady-state term q_{ss} that does not approach 0.

   c. Express the steady-state part of the response in the form $q_{ss} = D\cos(\omega t - \alpha)$. (See section 3.4 of the textbook. Your expressions for D and α will contain ω.)

   d. Plot D vs. ω on a loglog plot for $0.01 \leq \omega \leq 1000$. (Notice that the amplitude of the response decreases as ω increases. This means that the RC circuit acts as a low-pass filter, filtering out high-frequency signals.)