**MATH.2360 Engineering Differential Equations**
**Solutions to Sample Problems for Exam # 2**

**Problem 1.** Consider the autonomous differential equation $\dfrac{dx}{dt} = 4x^2 - x^4$.

a. Find all critical points (equilibrium solutions) of this d.e.

$4x^2 - x^4 = 0 \Rightarrow x^2\left(4 - x^2\right) = 0 \Rightarrow x^2(2+x)(2-x) = 0 \Rightarrow$
$\boxed{\text{the critical points are } -2, 0 \text{ and } 2}$

b. Draw the phase line (phase diagram) for this d.e.

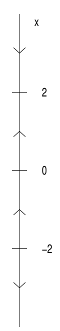
The three critical points divide the phase line into 4 intervals: $x > 2$, $0 < x < 2$, $-2 < x < 0$, and $x < -2$.

$\left.\dfrac{dx}{dt}\right|_{x=3} = 3^2(2+3)(2-3) < 0$, so the direction arrow points down for $x > 2$.

$\left.\dfrac{dx}{dt}\right|_{x=1} = 1^2(2+1)(2-1) > 0$, so the direction arrow points up for $0 < x < 2$.

$\left.\dfrac{dx}{dt}\right|_{x=-1} = (-1)^2(2-1)(2-(-1)) > 0$, so the arrow points up for $-2 < x < 0$.

$\left.\dfrac{dx}{dt}\right|_{x=-3} = (-3)^2(2-3)(2-(-3)) < 0$, so the arrow points down for $x < -2$.



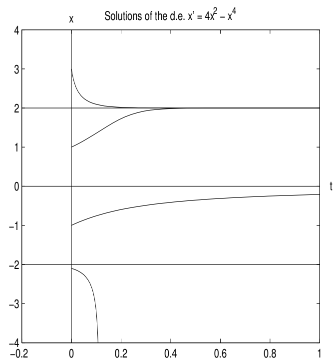

c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\text{2 is stable but } -2 \text{ and } 0 \text{ are unstable}}$.

d. If $x(0) = -1$, what value will $x(t)$ approach as $t$ increases?

Since $-1$ lies in the interval $-2 < x < 0$, we can see from the phase line that $\boxed{x(t) \to 0}$ as $t$ increases.

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes.




**Problem 2.** Solve the following differential equations:

a. $y'' - 10y' + 25y = 0$

Characteristic equation: $r^2 - 10r + 25 = 0 \Rightarrow (r-5)^2 = 0 \Rightarrow$
$r = 5$ (repeated root). Therefore, $\boxed{\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}}$

b. $y'' + y' - 2y = 0$

Characteristic equation: $r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow$
$r = -2$ or $r = 1$. Therefore, $\boxed{\boxed{y = c_1 e^{-2x} + c_2 e^{x}}}$

c. $y'' + 6y' + 8y = 0$

Characteristic equation: $r^2 + 6r + 8 = 0 \Rightarrow (r+4)(r+2) = 0 \Rightarrow$
$r = -4$ or $r = -2$. Therefore, $\boxed{\boxed{y = c_1 e^{-4x} + c_2 e^{-2x}}}$

d. $y'' - 6y' + 9y = 0$

Characteristic equation: $r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow$
$r = 3$ (repeated root). Therefore, $\boxed{\boxed{y = c_1 e^{3x} + c_2 x e^{3x}}}$

**Problem 3.** Solve the following initial value problem.

$$3x^2 + 6xy + \left[3x^2 + 2y\right]\frac{dy}{dx} = 0, \quad y(1) = 0.$$

$$\underbrace{3x^2 + 6xy}_{M} + \underbrace{\left[3x^2 + 2y\right]}_{N}\frac{dy}{dx} = 0.$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[3x^2 + 6xy\right] = 6x. \quad \dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[3x^2 + 2y\right] = 6x.$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. Therefore, the solution of the d.e. is $f(x, y) = c$, where the function $f$ satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 3x^2 + 6xy$ and $\dfrac{\partial f}{\partial y} = N = 3x^2 + 2y$. $\dfrac{\partial f}{\partial x} = 3x^2 + 6xy \Rightarrow$ $f = \displaystyle\int \left(3x^2 + 6xy\right)\, dx = x^3 + 3x^2y + g(y) \Rightarrow \dfrac{\partial f}{\partial y} = 3x^2 + g'(y)$.

But $\dfrac{\partial f}{\partial y} = N = 3x^2 + 2y$, so $3x^2 + g'(y) = 3x^2 + 2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2$.

Therefore, $f = x^3 + 3x^2y + y^2$, so the solution of the d.e. is $x^3 + 3x^2y + y^2 = c$.

The initial condition $y(1) = 0 \Rightarrow 1^3 + 3(1)^2(0) + 0^2 = c \Rightarrow c = 1$. Therefore, the solution of the given IVP is $\boxed{\boxed{x^3 + 3x^2y + y^2 = 1}}$

**Problem 4.** Solve the following initial value problem.

$$x^3 + y^3 - xy^2\frac{dy}{dx} = 0, \quad y(1) = 3.$$

$x^3 + y^3 - xy^2\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}$. Since $dy/dx$ equals a rational function in which each term has the same degree (3), the d.e. is homogeneous. We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace $y$ by $xv$:

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2} \Rightarrow v + x\frac{dv}{dx} = \frac{x^3 + (xv)^3}{x(xv)^2} = \frac{x^3\left(1 + v^3\right)}{x^3v} = \frac{1 + v^3}{v^2} = \frac{1}{v^2} + v \Rightarrow x\frac{dv}{dx} = \frac{1}{v^2}$$

$$\Rightarrow v^2\, dv = \frac{1}{x}\, dx \Rightarrow \int v^2\, dv = \int \frac{1}{x}\, dx \Rightarrow \frac{v^3}{3} = \ln(x) + c \Rightarrow \frac{(y/x)^3}{3} = \ln(x) + c \Rightarrow$$

$(y/x)^3 = 3\ln(x) + \underbrace{3c}_{c_1}$. The initial condition $y(1) = 3 \Rightarrow (3/1)^3 = 3\ln(1) + c_1 \Rightarrow c_1 = 27$ $\boxed{\text{1 pt.}}$.

Therefore, $(y/x)^3 = 3\ln(x) + 27 \Rightarrow (y/x) = [3\ln(x) + 27]^{1/3} \Rightarrow \boxed{\boxed{y = x\,[3\ln(x) + 27]^{1/3}}}$.

**Problem 5.** Solve the following initial value problem.

$$x^4 + 4y^4 - 4xy^3\frac{dy}{dx} = 0, \quad y(1) = 2.$$

$x^4 + 4y^4 - 4xy^3\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{x^4 + 4y^4}{4xy^3}$. $dy/dx$ equals a rational function, and every term has the same degree (4). Therefore, this d.e. is homogeneous.

We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace $y$ by $xv$:

$$\frac{dy}{dx} = \frac{x^4 + 4y^4}{4xy^3} \Rightarrow v + x\frac{dv}{dx} = \frac{x^4 + 4(xv)^4}{4x(xv)^3} = \frac{x^4\left(1 + 4v^4\right)}{4x^4v^3} = \frac{1 + 4v^4}{4v^3} = \frac{1}{4v^3} + v \Rightarrow x\frac{dv}{dx} = \frac{1}{4v^3}$$

$$\Rightarrow 4v^3\, dv = \frac{1}{x}\, dx \Rightarrow \int 4v^3\, dv = \int \frac{1}{x}\, dx \Rightarrow v^4 = \ln(x) + c \Rightarrow \left(\frac{y}{x}\right)^4 = \ln(x) + c$$

The initial condition $y(1) = 2 \Rightarrow (2/1)^4 = \ln(1) + c \Rightarrow c = 16$.

Therefore, $\left(\dfrac{y}{x}\right)^4 = \ln(x) + 16 \Rightarrow \dfrac{y}{x} = [\ln(x) + 16]^{1/4} \Rightarrow \boxed{\boxed{y = x\,[\ln(x) + 16]^{1/4}}}$.

**Problem 6. (20 points)** Solve the following initial value problem.

$$2xy^2 + 3x^2 + \left(2x^2y + 4y^3\right)\frac{dy}{dx} = 0, \quad y(1) = 2.$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{2xy^2 + 3x^2}_{M} + \underbrace{\left(2x^2y + 4y^3\right)}_{N}\frac{dy}{dx} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[2xy^2 + 3x^2\right] = 4xy. \quad \dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[2x^2y + 4y^3\right] = 4xy.$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. Therefore, the solution of the d.e. is $f(x, y) = c$, where the function $f$ satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 2xy^2 + 3x^2$ and $\dfrac{\partial f}{\partial y} = N = 2x^2y + 4y^3$. $\dfrac{\partial f}{\partial x} = 2xy^2 + 3x^2 \Rightarrow f = \int \left(2xy^2\right.$

$\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[x^2y^2 + x^3 + g(y)\right] = 2x^2y + g'(y)$

But $\dfrac{\partial f}{\partial y} = N = 2x^2y + 4y^3 \Rightarrow 2x^2y + g'(y) = 2x^2y + 4y^3 \Rightarrow g'(y) = 4y^3 \Rightarrow g(y) = y^4 \Rightarrow$
$f = x^2y^2 + x^3 + y^4$

Therefore, the solution of the d.e. is $x^2y^2 + x^3 + y^4 = c$
$y(1) = 2 \Rightarrow 1^22^2 + 1^3 + 2^4 = c \Rightarrow c = 21$.

Therefore, the solution of the initial value problem is $\boxed{\boxed{x^2y^2 + x^3 + y^4 = 21}}$

**Problem 7. (10 points)** Let $P$ denote the population of a colony of dodos. Suppose that the birth rate $\beta$ (number of births per week per dodo) equals 0 and that the death rate $\delta$ (number of deaths per week per dodo) is constant. Suppose the initial population is 100, and after ten weeks the population is 50. What is the population after 43 weeks?

$\dfrac{dP}{dt} = \beta P - \delta P = (0)P - (k)\,P = -kP.$

This is a separable d.e: $\dfrac{dP}{dt} = -kP \Rightarrow \dfrac{dP}{P} = -k\ dt$

$\Rightarrow \displaystyle\int P^{-1}\ dP = \int -k\ dt \Rightarrow \ln(P) = -kt + c \Rightarrow P = e^{-kt+c} = e^{-kt}\underbrace{e^c}_{c_1}.$

$P(0) = 100 \Rightarrow 100 = e^0c_1 \Rightarrow c_1 = 100$

$\Rightarrow P = 100e^{-kt}$. $P(10) = 50 \Rightarrow 50 = 100e^{-10k} \Rightarrow \dfrac{50}{100} = e^{-10k} \Rightarrow \ln(0.5) = \ln\left(e^{-10k}\right) = -10k \Rightarrow k = -\dfrac{\ln(0.5)}{10}.$

Therefore, $\boxed{\boxed{P(43) = 100e^{-43k} \approx 5 \text{ dodos}}}$.

**Problem 8. (10 points)** A ball of mass $m$ falling vertically downward experiences two forces: its weight, and a drag force proportional to the *square* of its velocity. Let $t$ denote time, and let $v$ denote the ball's velocity at time $t$. (Assume $v < 0$ for the falling object. In other words, up is the positive direction.)

**Write a differential equation ($dv/dt$ = something) modeling the ball's motion.** Make sure the drag force has the proper sign (+ or −). All parameters are assumed to be positive.

**DO NOT SOLVE THE DIFFERENTIAL EQUATION, JUST WRITE IT DOWN.**

Newton's Second Law says that $F = ma$, where $F$ is the total force acting on an object and $a$ is the object's acceleration.

The total force acting on the ball is $F = \underbrace{-mg}_{\text{weight}} + \underbrace{kv^2}_{\text{drag}}$

(Because the ball is falling, the drag is in the upward direction, so the drag force must be positive.)

Therefore, $F = ma \Rightarrow -mg + kv^2 = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = -g + \underbrace{\frac{k}{m}}_{k_1} v^2 \Rightarrow \boxed{\boxed{\frac{dv}{dt} = -g + k_1 v^2}}$