MATH.2360 Engineering Differential Equations
Solutions to Sample Problems for Exam # 3

**Problem 1.** Solve the following differential equations.

a. $y'' + 2y' + 2y = 0$.

Characteristic equation: $r^2 + 2r + 2 = 0 \Rightarrow$
$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm 1i$$
Therefore, $y = c_1 e^{-x}\cos(1x) + c_2 e^{-x}\sin(1x)$, or $\boxed{\boxed{y = c_1 e^{-x}\cos(x) + c_2 e^{-x}\sin(x)}}$

b. $y^{(3)} - 10y'' + 25y' = 0$.

Characteristic equation: $r^3 - 10r^2 + 25r = 0 \Rightarrow r\,(r^2 - 10r + 25) = 0 \Rightarrow r(r-5)^2 = 0 \Rightarrow$ $r = 0$ or $r = 5$ (double root). Therefore, $y = c_1 e^{0x} + c_2 e^{5x} + c_3 x e^{5x}$, or
$\boxed{\boxed{y = c_1 + c_2 e^{5x} + c_3 x e^{5x}}}$

c. $y'' - 6y' + 25y = 0$.

Characteristic equation: $r^2 - 6r + 25 = 0 \Rightarrow$
$$r = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(25)}}{2(1)} = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$$
Therefore, $\boxed{\boxed{y = c_1 e^{3x}\cos(4x) + c_2 e^{3x}\sin(4x)}}$

d. $y^{(4)} + 4y^{(3)} + 4y'' = 0$.

Characteristic equation: $r^4 + 4r^3 + 4r^2 = 0 \Rightarrow r^2(r^2 + 4r + 4) = 0 \Rightarrow r^2(r+2)^2 = 0 \Rightarrow$ $r = 0$ or $r = -2$ (both double roots). Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{-2x} + c_4 x e^{-2x}$, or
$\boxed{\boxed{y = c_1 + c_2 x + c_3 e^{-2x} + c_4 x e^{-2x}}}$

**Problem 2.** Solve the following initial value problem:
$$y'' + 2y' + y = 4x + 8e^x,\;\; y(0) = 2,\;\; y'(0) = -2.$$

Step 1. Find $y_c$ by solving the homogeneous d.e. $y'' + 2y' + y = 0$.
Characteristic equation: $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$ double root.
Therefore, $y_c = c_1 e^{-x} + c_2 x e^{-x}$.

Step 2. Find $y_p$.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $4x + 8e^x$ in the given d.e. is the sum of a polynomial of degree one and an exponential function, we should guess that $y_p$ is the sum of a polynomial of degree one and an exponential function: $y_p = Ax + B + Ce^x$. No term in this guess duplicates a term in $y_c$, so there is no need to modify

this guess. $y = Ax + B + Ce^x \Rightarrow y' = A + Ce^x \Rightarrow y'' = Ce^x$. Therefore, the left side of the d.e. is $y'' + 2y' + y = Ce^x + 2\left[A + Ce^x\right] + Ax + B + Ce^x = Ax + (B + 2A) + 4Ce^x$. We want this to equal the nonhomogeneous term $4x + 8e^x$: $Ax + (B + 2A) + 4Ce^x = 4x + 8e^x \Rightarrow A = 4,\ B + 2A = 0,\ 4C = 8 \Rightarrow A = 4,\ B = -8,\ C = 2$. Thus, $y_p = 4x - 8 + 2e^x$.

Method 2: Variation of Parameters. From $y_c$ we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-x}$ and $y_2 = xe^{-x}$. The Wronskian is given by

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-x}\left(e^{-x} - xe^{-x}\right) - \left(-e^{-x}\right)xe^{-x} = e^{-2x}$.

$u_1 = \int \dfrac{-y_2 f(x)}{W(x)}\, dx = -\int \dfrac{xe^{-x}\left(4x + 8e^x\right)}{e^{-2x}}\, dx = -\int \left[4x^2e^x + 8xe^{2x}\right]\, dx =$
$-4\left[x^2e^x - 2\int xe^x\, dx\right] - 2\int (2x)e^{2x}\, d(2x) = -4\left[x^2e^x - 2(x-1)e^x\right] - 2(2x-1)e^{2x} =$
$\left(-4x^2 + 8x - 8\right)e^x + (-4x + 2)\,e^{2x}$ using formulas 46 and 47 from the table of integrals.

$u_2 = \int \dfrac{y_1 f(x)}{W(x)}\, dx = \int \dfrac{e^{-x}\left(4x + 8e^x\right)}{e^{-2x}}\, dx = \int \left[4xe^x + 8e^{2x}\right]\, dx = 4\left[(x-1)e^x\right] + 4e^{2x}$ using formula 46 from the table of integrals.

Therefore, $y_p = u_1y_1 + u_2y_2 = \left[\left(-4x^2 + 8x - 8\right)e^x + (-4x + 2)\,e^{2x}\right]e^{-x} + \left[4\left((x-1)e^x\right) + 4e^{2x}\right]xe^{-x} =$
$-4x^2 + 8x - 8 + (-4x + 2)\,e^x + 4(x-1)x + 4xe^x = 4x - 8 + 2e^x$

Step 3. $y = y_c + y_p$, so $y = c_1e^{-x} + c_2xe^{-x} + 4x - 8 + 2e^x$.

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.
$y = c_1e^{-x} + c_2xe^{-x} + 4x - 8 + 2e^x \Rightarrow y' = -c_1e^{-x} + c_2\left[e^{-x} - xe^{-x}\right] + 4 + 2e^x$.
$y(0) = 2 \Rightarrow 2 = c_1e^0 + c_2(0)e^0 + 4(0) - 8 + 2e^0 = c_1 - 6 \Rightarrow c_1 = 8$
$y'(0) = -2 \Rightarrow -2 = -c_1e^0 + c_2\left[e^0 - (0)e^0\right] + 4 + 2e^0 = -c_1 + c_2 + 6 \Rightarrow c_2 = c_1 - 8 = 0$

Therefore, $\boxed{\boxed{y = 8e^{-x} + 4x - 8 + 2e^x}}$

**Problem 3.** Solve the following initial value problem:

$$y'' - 2y' = 15\sin(x),\ y(0) = 0,\ y'(0) = -3.$$

Step 1. Find $y_c$ by solving the homogeneous d.e. $y'' - 2y' = 0$.
Characteristic equation: $r^2 - 2r = 0 \Rightarrow r(r-2) = 0 \Rightarrow r = 0$ or $r = 2$.
Therefore, $y_c = c_1e^{0x} + c_2e^{2x} = c_1 + c_2e^{2x}$.

Step 2. Find $y_p$.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $15\sin(x)$ in the given d.e. is a sine function, we should guess that $y_p$ is a combination of a cosine function and a sine function with the same coefficient of $x$ as in the nonhomogeneous term: $y_p = A\cos(x) + B\sin(x)$. No term in this guess duplicates a term in $y_c$, so there is no need to modify the guess. $y = A\cos(x) + B\sin(x) \Rightarrow y' = -A\sin(x) + B\cos(x) \Rightarrow y'' = -A\cos(x) - B\sin(x)$. Therefore, the left side of the d.e. is $y'' - 2y' = -A\cos(x) - B\sin(x) - 2\left[-A\sin(x) + B\cos(x)\right] = \left[-A - 2B\right]\cos(x) + \left[2A - B\right]\sin(x)$. We want this to equal the nonhomogeneous term $15\sin(x)$ so $2A - B = 15,\ -A - 2B = 0 \Rightarrow A = 6,\ B = -3$. Thus, $y_p = 6\cos(x) - 3\sin(x)$.

Method 2: Variation of Parameters. From $y_c$ we obtain two independent solutions of the homogeneous d.e: $y_1 = 1$ and $y_2 = e^{2x}$. The Wronskian is given by

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = (1)\left(2e^{2x}\right) - (0)\,e^{-2x} = 2e^{2x}.$

$u_1 = \displaystyle\int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{2x}\,(15\sin(x))}{2e^{2x}}\,dx = -\frac{15}{2}\int \sin(x)\,dx = \frac{15}{2}\cos(x)$

$u_2 = \displaystyle\int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{(1)\,(15\cos(x))}{2e^{2x}}\,dx = \frac{15}{2}\int \left[e^{-2x}\sin(x)\right]\,dx = \frac{15}{2}\left\{\frac{e^{-2x}}{(-2)^2+1^2}\left[-2\sin(x) - \sin(x)\right]\right\}$

$= -\dfrac{15}{2}e^{-2x}\left[2\sin(x)+\cos(x)\right]$ using formula 49 from the Table of Integrals.

Therefore, $y_p = u_1y_1 + u_2y_2 = \left[\dfrac{15}{2}\cos(x)\right](1) + \left\{-\dfrac{3}{2}e^{-2x}\left[2\sin(x)+\cos(x)\right]\right\}e^{2x} =$

$\dfrac{15}{2}\cos(x) - 3\sin(x) - \dfrac{3}{2}\cos(x) = 6\cos(x) - 3\sin(x).$

Step 3. $y = y_c + y_p$, so $y = c_1 + c_2e^{2x} + 6\cos(x) - 3\sin(x)$.

Step 4. Use the initial conditions to determine the value of $c_1$ and $c_2$.
$y = c_1 + c_2e^{2x} + 6\cos(x) - 3\sin(x) \Rightarrow y' = 2c_2e^{2x} - 6\sin(x) - 3\cos(x)$.
$y(0) = 0 \Rightarrow 0 = c_1 + c_2e^0 + 6\cos(0) - 3\sin(0) = c_1 + c_2 + 6 \Rightarrow c_1 + c_2 = -6$.
$y'(0) = -3 \Rightarrow -3 = 2c_2e^0 - 6\sin(0) - 3\cos(0) = 2c_2 - 3 \Rightarrow c_2 = 0$. $c_1 + c_2 = -6 \Rightarrow c_1 = -6 - c_2 = -6$.
Therefore, $\boxed{\boxed{y = -6 + 6\cos(x) - 3\sin(x)}}$

**Problem 4.** Consider an RLC circuit with inductance $H = 1$ henry, resistance $R = 2\Omega$, capacitance $C = 1/16$ farad, and applied voltage $E(t) = 32\cos(4t)$ volts. Find the steady periodic current $I_{sp}(t)$.

The d.e. describing an RLC circuit is $LQ'' + RQ' + \dfrac{Q}{C} = E(t)$.
In this problem, the d.e. becomes $Q'' + 2Q' + 16Q = 32\cos(4t)$.

The steady periodic solution is the particular solution. Since the nonhomogeneous term $32\cos(4t)$ is a cosine, we should guess that $Q_p$ is a combination of a cosine and a sine with the same frequency: $Q_p = A\cos(4t) + B\sin(4t)$. (No part of this guess will duplicate part of $Q_c$ because $Q_c$ is a transient term containing decaying exponential functions.)
$Q = A\cos(4t) + B\sin(4t) \Rightarrow Q' = -4A\sin(4t) + 4B\cos(4t) \Rightarrow$
$Q'' = -16A\cos(4t) - 16B\sin(4t)$. Therefore, the left side of the d.e. is
$Q'' + 2Q' + 16Q = -16A\cos(4t) - 16B\sin(4t) + 2\left[-4A\sin(4t) + 4B\cos(4t)\right] + 16\left[A\cos(4t) + B\sin(4t)\right]$
$= 8B\cos(4t) - 8A\sin(4t)$.
We want this to equal the nonhomogeneous term $32\cos(4t)$:
$8B\cos(4t) - 8A\sin(4t) = 32\cos(4t) \Rightarrow 8B = 32,\ -8A = 0 \Rightarrow A = 0$ and $B = 4$. Therefore,
$Q_{\mathrm{sp}} = 4\sin(4t)$. Current is the derivative of $Q$, so $\boxed{\boxed{I_{\mathrm{sp}} = 16\cos(4t)}}$

**Problem 5.** Consider a free (unforced), damped mass-spring system with mass $m = 1$ kg, damping constant $c = 6$ N·s/m, and spring constant $k = 10$ N/m. Assume that $x(0) = -1$ and $x'(0) = 0$.

a. Find the position function $x(t)$.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_e(t)$.
In this problem, the d.e. becomes $x'' + 6x' + 10x = 0$.

The characteristic equation is $r^2 + 6r + 10 = 0 \Rightarrow$
$r = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)} = \dfrac{-6 \pm \sqrt{-4}}{2} = \dfrac{-6 \pm 2i}{2} = -3 \pm i$
Therefore, $x = c_1 e^{-3t}\cos(t) + c_2 e^{-3t}\sin(t)$ so $x' = c_1\left[-3e^{-t}\cos(t) - e^{-3t}\sin(t)\right] + c_2\left[-3e^{-t}\sin(t) + e^{-3t}\right.$
$x(0) = -1 \Rightarrow -1 = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = c_1 \Rightarrow c_1 = -1.$
$x'(0) = 0 \Rightarrow 0 = c_1\left[-3e^0\cos(0) - e^0\sin(0)\right] + c_2\left[-3e^0\sin(0) + e^0\cos(0)\right] = -3c_1 + c_2 \Rightarrow c_2 = 3c_1 = -3$
Therefore $\boxed{x = -e^{-3t}\cos(t) - 3e^{-3t}\sin(t)}$

b. Express your solution from part a in the form $x = Ce^{-pt}\cos(\omega_1 t - \alpha)$

$C = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$
Because $c_1 < 0$ we have $\alpha = \pi + \tan^{-1}(c_2/c_1) = \pi + \tan^{-1}(3)$
Therefore, $\boxed{x = \sqrt{10}e^{-3t}\cos\left(t - \left(\pi + \tan^{-1}(3)\right)\right)}$

c. Is this system overdamped, underdamped, or critically damped?

The characteristic equation has complex roots, so $\boxed{\text{the system is underdamped.}}$

**Problem 6.** Solve the system $\begin{cases} x' &= 2x + y \\ y' &= 3x \end{cases}$

Note: $x' = dx/dt$ and $y' = dy/dt$. $t$ is the independent variable.

Take the derivative of both sides of the second d.e. in the system: $y' = 3x \Rightarrow y'' = 3x'$. The first d.e. in the system is $x' = 2x + y$. Therefore, $y'' = 3(2x + y) = 6x + 3y$. From the second d.e. in the system, $6x = 2y'$, so we have $y'' = 2y' + 3y$

$y'' = 2y' + 3y \Rightarrow y'' - 2y' - 3y = 0.$
Characteristic equation: $r^2 - 2r - 3 = 0 \Rightarrow (r+1)(r-3) = 0 \Rightarrow r = -1$ or $r = 3 \Rightarrow y = c_1 e^{-t} + c_2 e^{3t}$.

The second d.e. in the given system says $x = y'/3$, so $x = \left(-c_1 e^{-t} + 3c_2 e^{3t}\right)/3 = -\dfrac{1}{3}c_1 e^{-t} + c_2 e^{3t}$.
Therefore, the solution of the given system is

$$\boxed{x = -\frac{1}{3}c_1 e^{-t} + c_2 e^{3t}, \quad y = c_1 e^{-t} + c_2 e^{3t}}$$

**Problem 7.** Solve the system $\begin{cases} x' &= y \\ y' &= -3x - 4y \end{cases}$

Note: $x' = dx/dt$ and $y' = dy/dt$. $t$ is the independent variable.

Take the derivative of both sides of the first d.e. in the system: $x' = y \Rightarrow x'' = y'$. The second d.e. in the system is $y' = -3x - 4y$. Therefore, $x'' = -3x - 4y$. From the first d.e. in the system, $y = x'$, so we have $x'' = -3x - 4x'$

$x'' = -3x - 4x' \Rightarrow x'' + 4x' + 3x = 0.$
Characteristic equation: $r^2 + 4r + 3 = 0 \Rightarrow (r+1)(r+3) = 0 \Rightarrow r = -1$ or $r = -3 \Rightarrow x = c_1 e^{-t} + c_2 e^{-3t}$.

The first d.e. in the given system says $y = x'$, so $y = -c_1 e^{-t} - 3c_2 e^{-3t}$. Therefore, the solution of the given system is

$$\boxed{\boxed{x = c_1 e^{-t} + c_2 e^{-3t},\ y = -c_1 e^{-t} - 3c_2 e^{-3t}}}$$