92.236 Engineering Differential Equations Exam # 2 Solutions Fall 2017

Problem 1. 20 points Consider the autonomous differential equation $\dfrac{dx}{dt} = -x^2 - x + 6$.

a. Find all critical points (equilibrium solutions) of this d.e.

$-x^2 - x + 6 \Rightarrow -\left(x^2 + x - 6\right) = 0 \Rightarrow -(x+3)(x-2) = 0 \Rightarrow$
$\boxed{\boxed{\text{the equilibrium solutions are } x = -3 \text{ and } x = 2}}$ $\boxed{\text{3 pts.}}$

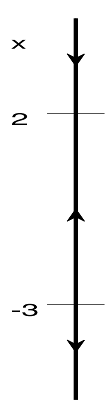
b. Draw the phase line (phase diagram) for this d.e. $\boxed{\text{8 pts.}}$

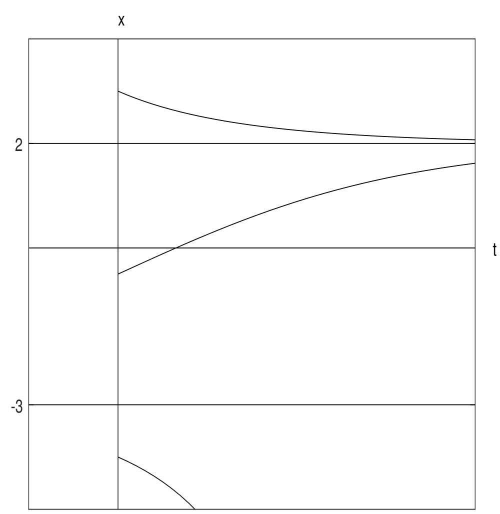
The two equilibrium solutions divide the phase line into 3 intervals: $x > 2$, $-3 < x < 2$, and $x < -3$.

$\left.\dfrac{dx}{dt}\right|_{x=3} = -(3+3)(3-2) < 0$, so the direction arrow points down for $x > 2$.

$\left.\dfrac{dx}{dt}\right|_{x=0} = -(0+4)(0-2) > 0$, so the direction arrow points up for $-3 < x < 2$.

$\left.\dfrac{dx}{dt}\right|_{x=-4} = -(-4+3)(-4-2) < 0$, so the direction arrow points down for $x < -3$.





c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\boxed{2 \text{ is stable and } -3 \text{ is unstable}}}$. $\boxed{\text{2 pts.}}$

d. If $x(0) = 0$, what value will $x(t)$ approach as t increases?

Since 0 lies in the interval $-3 < x < 2$, we can see from the phase line that $\boxed{\boxed{x(t) \to 2}}$ as t increases. $\boxed{\text{3 pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. $\boxed{\text{4 pts.}}$

See the figure above.

Problem 2. (20 points) Solve the following differential equations:

a. $y'' + 4y' + 4y = 0$

Characteristic equation: $r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow$
$r = -2$ (repeated root). $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-2x} + c_2 x e^{-2x}}}$ $\boxed{\text{6 pts.}}$

b. $y'' - 3y' - 4y = 0$

Characteristic equation: $r^2 - 3r - 4 = 0 \Rightarrow (r+1)(r-4) = 0 \Rightarrow$
$r = -1$ or $r = 4$. $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-x} + c_2 e^{4x}}}$ $\boxed{\text{6 pts.}}$

Problem 3. (20 points) Solve the following initial value problem.

$$4x^3 + 3y^3 - 3xy^2 \frac{dy}{dx} = 0, \quad y(1) = 2$$

$4x^3 + 3y^3 - 3xy^2 \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{4x^3 + 3y^3}{3xy^2}$. dy/dx equals a rational function, and every term has the same degree (3). Therefore, this d.e. is homogeneous. $\boxed{\text{4 pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{4x^3 + 3y^3}{3xy^2} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{4x^3 + 3(xv)^3}{3x(xv)^2}}_{\boxed{\text{4 pts.}}} = \frac{x^3\left(4 + 3v^3\right)}{3x^3v^2} = \frac{4}{3v^2} + v \Rightarrow \underbrace{x\frac{dv}{dx} = \frac{4}{3v^2}}_{\boxed{\text{3 pts.}}}$$

$$\underbrace{\Rightarrow 3v^2\, dv = \frac{4}{x}\, dx}_{\boxed{\text{2 pts.}}} \Rightarrow \int 3v^2\, dv = \int \frac{4}{x}\, dx \Rightarrow \underbrace{v^3 = 4\ln(x) + c}_{\boxed{\text{3 pts.}}} \Rightarrow \underbrace{\left(\frac{y}{x}\right)^3 = 4\ln(x) + c}_{\boxed{\text{2 pts.}}}$$

The initial condition $y(1) = 2 \Rightarrow (2/1)^3 = 4\ln(1) + c \Rightarrow c = 8$ $\boxed{\text{2 pts.}}$.

Therefore, $\left(\dfrac{y}{x}\right)^3 = 4\ln(x) + 8 \Rightarrow \boxed{\boxed{y = x\left(4\ln(x) + 8\right)^{1/3}}}$.

Problem 4. (20 points) Solve the following initial value problem.

$$4x^3 + 3y^3 + \left[9xy^2 - 6y\right]\frac{dy}{dx} = 0, \quad y(2) = 1$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{4x^3 + 3y^3}_{M} + \underbrace{\left(9xy^2 - 6y\right)}_{N}\frac{dy}{dx} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[4x^3 + 3y^3\right] = 9y^2.$ $\boxed{\text{1 pt.}}$ $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[9xy^2 - 6y\right] = 9y^2.$ $\boxed{\text{1 pt.}}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{3 pts.}}$ Therefore, the solution of the d.e. is $f(x,y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 4x^3 + 3y^3$ and $\dfrac{\partial f}{\partial y} = N = 9xy^2 - 6y$.

$\dfrac{\partial f}{\partial x} = 4x^3 + 3y^3 \Rightarrow f = \displaystyle\int \left(4x^3 + 3y^3\right)\,\partial x = x^4 + 3xy^3 + g(y)$ $\boxed{\text{6 pts.}}$

$\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[x^4 + 3xy^3 + g(y)\right] = 9xy^2 + g'(y)$

But $\dfrac{\partial f}{\partial y} = N = 9xy^2 - 6y \Rightarrow 9xy^2 + g'(y) = 9xy^2 - 6y \Rightarrow g'(y) = -6y \Rightarrow g(y) = -3y^2 \Rightarrow$

$f = x^4 + 3xy^3 - 3y^2$ $\boxed{\text{6 pts.}}$

Therefore, the solution of the d.e. is $x^4 + 3xy^3 - 3y^2 = c$ $\boxed{\text{2 pts.}}$

$y(2) = 1 \Rightarrow 2^4 + 3(2)(1)^3 - 3(1)^2 = c \Rightarrow c = 19.$ $\boxed{\text{1 pt.}}$

Therefore, the solution of the initial value problem is $\boxed{\boxed{x^4 + 3xy^3 - 3y^2 = 19}}$

Problem 5. (10 points) Let P denote the population of a colony of tribbles. Suppose that the birth rate β (number of births per week per tribble) is proportional to $1/\sqrt{P}$ and that the death rate δ (number of deaths per week per tribble) equals 0. Suppose the initial population is 16, and after one week the population is 25. What is the population after 3 weeks?

$\dfrac{dP}{dt} = \beta P - \delta P = \left(\dfrac{k}{\sqrt{P}}\right)P - (0)\,P = k\sqrt{P}.$ $\boxed{\text{3 pts.}}$

This is a separable d.e: $\dfrac{dP}{dt} = k\sqrt{P} \Rightarrow \dfrac{dP}{\sqrt{P}} = k\ dt \Rightarrow \displaystyle\int P^{-1/2}\,dP = \int k\ dt \Rightarrow 2P^{1/2} = kt + c.$

$\boxed{\text{4 pts.}}$

$P(0) = 16 \Rightarrow 2(16)^{1/2} = k(0) + c \Rightarrow c = 8 \Rightarrow 2P^{1/2} = kt + 8$ $\boxed{\text{1 pt.}}$

$P(1) = 25 \Rightarrow 2(25)^{1/2} = k(1) + 8 \Rightarrow 10 = k + 8 \Rightarrow k = 2.$ $\boxed{\text{1 pt.}}$

Therefore, $2\left(P(3)\right)^{1/2} = 2(3) + 8 = 14 \Rightarrow \boxed{\boxed{P(3) = 49 \text{ tribbles}}}$. $\boxed{\text{1 pt.}}$