Problem 1. (20 pts.) Solve the following differential equations.

a. (8 pts.) $y'' + 6y' + 10y = 0$

Characteristic equation: $r^2 + 6r + 10 = 0 \Rightarrow r = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)} = \dfrac{-6 \pm \sqrt{-4}}{2} = \dfrac{-6 \pm 2i}{2} = -3 \pm i$

$\boxed{4 \text{ pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-3x}\cos(x) + c_2 e^{-3x}\sin(x)}}$ $\boxed{4 \text{ pts.}}$

b. (12 pts.) $y^{(4)} - 4y'' = 0.$

Characteristic equation: $r^4 - 4r^2 = 0 \Rightarrow r^2\left(r^2 - 4\right) = 0 \Rightarrow r^2(r+2)(r-2) = 0 \Rightarrow$
$r = 0$ (double root) or $r = -2$ or $r = 2$. $\boxed{4 \text{ pts.}}$ Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{-2x} + c_4 e^{2x}$,
or $\boxed{\boxed{y = c_1 + c_2 x + c_3 e^{-2x} + c_4 e^{2x}}}$ $\boxed{8 \text{ pts.}}$

Problem 2. (25 pts.) Solve the following initial value problem:

$$y'' + 5y' + 4y = 100xe^x,\ y(0) = -4,\ y'(0) = 0.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' + 5y' + 4y = 0$.
Characteristic equation: $r^2 + 5r + 4 = 0 \Rightarrow (r+4)(r+1) = 0 \Rightarrow r = -4$ or $r = -1$.
Therefore, $y_c = c_1 e^{-4x} + c_2 e^{-x}$. $\boxed{5 \text{ pts.}}$

Step 2. Find y_p.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $100xe^x$ in the given d.e. is a polynomial of degree 1 times an exponential function, we should guess that y_p is a polynomial of degree 1 times an exponential function:
$y_p = (Ax + B)\,e^x$. $\boxed{4 \text{ pts.}}$ No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{2 \text{ pts.}}$ $y = (Ax+B)\,e^x \Rightarrow y' = Ae^x + (Ax+B)\,e^x = (Ax + A + B)\,e^x \Rightarrow$
$y'' = Ae^x + (Ax + A + B)\,e^x = (Ax + 2A + B)\,e^x$. Therefore, the left side of the d.e. is
$y'' + 5y' + 4y = (Ax + 2A + B)\,e^x + 5\left[(Ax + A + B)\,e^x\right] + 4\left[(Ax+B)\,e^x\right] = (10Ax + 7A + 10B)\,e^x$.
We want this to equal the nonhomogeneous term $100xe^x$:
$(10Ax + 7A + 10B)\,e^x = 100xe^x \Rightarrow 10A = 100,\ 7A + 10B = 0 \Rightarrow A = 10,\ B = -7$. Thus, $y_p = (10x - 7)\,e^x$. $\boxed{9 \text{ pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-4x}$ and $y_2 = e^{-x}$. $\boxed{1 \text{ pt.}}$ The Wronskian is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-4x} & e^{-x} \\ -4e^{-4x} & -e^{-x} \end{vmatrix} = e^{-4x}\left(-e^{-x}\right) - \left(-4e^{-4x}\right)e^{-x} = 3e^{-5x}. \ \boxed{1 \text{ pt.}}$$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{-x}\left(100xe^x\right)}{3e^{-5x}}\,dx = -\frac{100}{3}\int \left[xe^{5x}\right]\,dx = -\frac{100}{3}\left[\frac{1}{25}(5x-1)\,e^{5x}\right] = -\frac{4}{3}(5x-1)\,e^{5x}$$

using formula 46 from the integral table, with $u = 5x$. $\boxed{4 \text{ pts.}}$

$$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-4x}\left(100xe^x\right)}{3e^{-5x}}\,dx = \frac{100}{3}\int \left[xe^{2x}\right]\,dx = \frac{100}{3}\left[\frac{1}{4}(2x-1)\,e^{2x}\right] = \frac{25}{3}(2x-1)\,e^{2x}$$

using formula 46 from the integral table, with $u = 2x$. $\boxed{4 \text{ pts.}}$

Therefore, $y_p = u_1 y_1 + u_2 y_2 = \left[-\dfrac{4}{3}(5x-1)\,e^{5x}\right]e^{-4x} + \left[\dfrac{25}{3}(2x-1)\,e^{2x}\right]e^{-x} = (10x-7)\,e^x$ $\boxed{5 \text{ pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1 e^{-4x} + c_2 e^{-x} (10x - 7) e^x$. $\boxed{\text{3 pts.}}$

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.

$y = y = c_1 e^{-4x} + c_2 e^{-x} + (10x - 7) e^x \Rightarrow y' = -4c_1 e^{-4x} - c_2 e^{-x} + 10e^x + (10x - 7) e^x = -4c_1 e^{-4x} - c_2 e^{-x} + (10x + 3) e^x$.

$y(0) = 4 \Rightarrow -4 = c_1 e^0 + c_2 e^0 - 7e^0 = c_1 + c_2 - 7 \Rightarrow c_1 + c_2 = 3$

$y'(0) = 0 \Rightarrow 0 = -4c_1 e^0 - c_2 e^0 + 3e^0 = -4c_1 - c_2 + 3 \Rightarrow -4c_1 - c_2 = -3. c_1 + c_2 = 3, \ -4c_1 - c_2 = -3 \Rightarrow c_1 = 0, \ c_2 = 3$ $\boxed{\text{2 pts.}}$ Therefore, $\boxed{\boxed{y = 3e^{-x} + (10x - 7) e^x}}$

Problem 3. (20 points) Consider a free, undamped mass-spring system with mass $m = 1$ kg and spring constant $k = 9$ N/m. Suppose $x(0) = -1$ and $x'(0) = -3$.

a. Find the position function $x(t)$.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_e(t)$. $\boxed{\text{3 pts.}}$

In this problem, $c = 0$ and $F_e(t) = 0$ so the d.e. becomes $x'' + 9x = 0$. $\boxed{\text{3 pts.}}$

The characteristic equation is $r^2 + 9 = 0$ so $r^2 = -9 \Rightarrow r = \pm 3i \Rightarrow x = c_1 \cos(3t) + c_2 \sin(3t)$ $\boxed{\text{10 pts.}}$

$x(0) = -1 \Rightarrow -1 = c_1 \cos(0) + c_2 \sin(0) = c_1$ $\boxed{\text{1 pt.}}$ so $x = -\cos(3t) + c_2 \sin(3t) \Rightarrow x' = 3\sin(3t) + 3c_2 \cos(3t)$.

$x'(0) = -3 \Rightarrow -3 = 3\sin(0) + 3c_2 \cos(0) = 3c_2 \Rightarrow c_2 = -1$. $\boxed{\text{1 pt.}}$ Therefore, $\boxed{\boxed{x = -\cos(3t) - \sin(3t)}}$

b. Express your solution from part a in the form $x = C \cos(\omega_0 t - \alpha)$.

$x = c_1 \cos(3t) + c_2 \sin(3t)$ where $c_1 = -1$ and $c_2 = -1$

$C = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ $\boxed{\text{1 pt.}}$

Because $c_1 < 0$ we have $\alpha = \pi + \tan^{-1}(c_2/c_1) = \pi + \tan^{-1}((-1)/(-1)) = \pi + \pi/4 = 5\pi/4$. $\boxed{\text{1 pt.}}$

Therefore, $\boxed{\boxed{x = \sqrt{2} \cos(3t - 5\pi/4)}}$

Problem 4. (20 points) Consider an RLC circuit with inductance $L = 1$ henry, resistance $R = 10\Omega$, capacitance $C = 1/9$ farad, and applied voltage $E(t) = 60\cos(3t)$ volts. Find the steady periodic current $I_{sp}(t)$.

The d.e. describing an RLC circuit is $LQ'' + RQ' + \dfrac{Q}{C} = E(t)$. $\boxed{\text{2 pts.}}$

In this problem, the d.e. becomes $Q'' + 10Q' + 9Q = 60\cos(3t)$. $\boxed{\text{1 pt.}}$

The steady periodic solution is the particular solution. $\boxed{\text{3 pts.}}$ Since the nonhomogeneous term $60\cos(3t)$ is a cosine, we should guess that Q_p is a combination of a cosine and a sine with the same frequency: $Q_p = A\cos(3t) + B\sin(3t)$. $\boxed{\text{5 pts.}}$ (No part of this guess will duplicate part of Q_c because Q_c is a transient term containing decaying exponential functions.)

$Q = A\cos(3t) + B\sin(3t) \Rightarrow Q' = -3A\sin(3t) + 3B\cos(3t) \Rightarrow$

$Q'' = -9A\cos(3t) - 9B\sin(3t)$. Therefore, the left side of the d.e. is

$Q'' + 10Q' + 9Q = -9A\cos(3t) - 9B\sin(3t) + 10[-3A\sin(3t) + 3B\cos(3t)] + 9[A\cos(3t) + B\sin(3t)]$

$= 30B\cos(3t) - 30A\sin(3t)$.

We want this to equal the nonhomogeneous term $60\cos(3t)$:

$30B\cos(3t) - 30A\sin(3t) = 60\cos(3t) \Rightarrow 30B = 60, \ -30A = 0 \Rightarrow A = 0$ and $B = 2$. Therefore, $Q_{\text{sp}} = 2\sin(3t)$. $\boxed{\text{8 pts.}}$ Current is the derivative of Q, so $\boxed{\boxed{I_{\text{sp}} = 6\cos(3t)}}$ $\boxed{\text{1 pt.}}$