92.236 Engineering Differential Equations   Exam # 2 Solutions   Fall 2017

Problem 1. 20 points Consider the autonomous differential equation $\dfrac{dx}{dt} = -x^2 - 2x + 8$.

a. Find all critical points (equilibrium solutions) of this d.e.

$-x^2 - 2x + 8 = 0 \Rightarrow -(x^2 + 2x - 8) = 0 \Rightarrow -(x+4)(x-2) = 0 \Rightarrow$
$\boxed{\boxed{\text{the equilibrium solutions are } x = -4 \text{ and } x = 2}}$ $\boxed{\text{3 pts.}}$

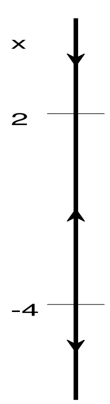
b. Draw the phase line (phase diagram) for this d.e. $\boxed{\text{8 pts.}}$

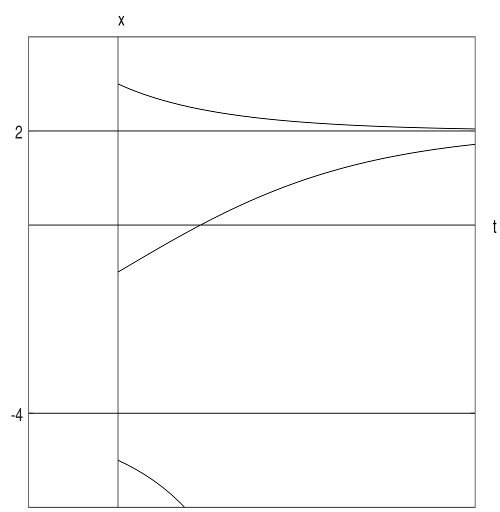
The two equilibrium solutions divide the phase line into 3 intervals: $x > 2$, $-4 < x < 2$, and $x < -4$.

$\left.\dfrac{dx}{dt}\right|_{x=3} = -(3+4)(3-2) < 0$, so the direction arrow points down for $x > 2$.

$\left.\dfrac{dx}{dt}\right|_{x=0} = -(0+4)(0-2) > 0$, so the direction arrow points up for $0 < x < 2$.

$\left.\dfrac{dx}{dt}\right|_{x=-5} = -(-5+4)(-5-2) < 0$, so the direction arrow points down for $x < -4$.





c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\boxed{2 \text{ is stable and } -4 \text{ is unstable}}}$. $\boxed{\text{2 pts.}}$

d. If $x(0) = 0$, what value will $x(t)$ approach as t increases?

Since 0 lies in the interval $-4 < x < 2$, we can see from the phase line that $\boxed{\boxed{x(t) \to 2}}$ as t increases. $\boxed{\text{3 pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. $\boxed{\text{4 pts.}}$

See the figure above.

Problem 2. (20 points) Solve the following differential equations:

a. $y'' + 8y' + 16y = 0$

Characteristic equation: $r^2 + 8r + 16 = 0 \Rightarrow (r+4)^2 = 0 \Rightarrow$
$r = -4$ (repeated root). $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-4x} + c_2 x e^{-4x}}}$ $\boxed{\text{6 pts.}}$

b. $y'' + 8y' + 12y = 0$

Characteristic equation: $r^2 + 8r + 12 = 0 \Rightarrow (r+6)(r+2) = 0 \Rightarrow$
$r = -6$ or $r = -2$. $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-6x} + c_2 e^{-2x}}}$ $\boxed{\text{6 pts.}}$

Problem 3. (20 points) Solve the following initial value problem.

$$x^2 + 2y^2 - 2xy\frac{dy}{dx} = 0, \quad y(1) = 2$$

$x^2 + 2y^2 - 2xy\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x^2 + 2y^2}{2xy}$. dy/dx equals a rational function, and every term has the same degree (2). Therefore, this d.e. is homogeneous. $\boxed{\text{4 pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\frac{dy}{dx}$ by $v + x\frac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{x^2 + 2y^2}{2xy} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{x^2 + 2(xv)^2}{2x(xv)}}_{\boxed{\text{4 pts.}}} = \frac{x^2(1 + 2v^2)}{2x^2 v} = \frac{1}{2v} + v \Rightarrow \underbrace{x\frac{dv}{dx} = \frac{1}{2v}}_{\boxed{\text{3 pts.}}}$$

$$\underbrace{\Rightarrow 2v\, dv = \frac{1}{x}\, dx}_{\boxed{\text{2 pts.}}} \Rightarrow \int 2v\, dv = \int \frac{1}{x}\, dx \Rightarrow \underbrace{v^2 = \ln(x) + c}_{\boxed{\text{3 pts.}}} \Rightarrow \underbrace{\left(\frac{y}{x}\right)^2 = \ln(x) + c}_{\boxed{\text{2 pts.}}}$$

The initial condition $y(1) = 2 \Rightarrow (2/1)^2 = \ln(1) + c \Rightarrow c = 4$ $\boxed{\text{2 pts.}}$.

Therefore, $\left(\frac{y}{x}\right)^2 = \ln(x) + 4 \Rightarrow \boxed{\boxed{y = x\sqrt{\ln(x) + 4}}}$.

Problem 4. (20 points) Solve the following initial value problem.

$$x^2 + 2y^2 + [4xy - 6y]\,\frac{dy}{dx} = 0, \quad y(3) = 1$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{x^2 + 2y^2}_{M} + \underbrace{(4xy - 6y)}_{N}\,\frac{dy}{dx} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[x^2 + 2y^2\right] = 4y.$ $\boxed{\text{1 pt.}}$ $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}[4xy - 6y] = 4y.$ $\boxed{\text{1 pt.}}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{3 pts.}}$ Therefore, the solution of the d.e. is $f(x, y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = x^2 + 2y^2$ and $\dfrac{\partial f}{\partial y} = N = 4xy - 6y$.

$\dfrac{\partial f}{\partial x} = x^2 + 2y^2 \Rightarrow f = \displaystyle\int \left(x^2 + 2y^2\right)\,\partial x = \frac{x^3}{3} + 2xy^2 + g(y)$ $\boxed{\text{6 pts.}}$

$\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[\dfrac{x^3}{3} + 2xy^2 + g(y)\right] = 4xy + g'(y)$

But $\dfrac{\partial f}{\partial y} = N = 4xy - 6y \Rightarrow 4xy + g'(y) = 4xy - 6y \Rightarrow g'(y) = -6y \Rightarrow g(y) = -3y^2 \Rightarrow$

$f = \dfrac{x^3}{3} + 2xy^2 - 3y^2$ $\boxed{\text{6 pts.}}$

Therefore, the solution of the d.e. is $\dfrac{x^3}{3} + 2xy^2 - 3y^2 = c$ $\boxed{\text{2 pts.}}$

$y(3) = 1 \Rightarrow \dfrac{3^3}{3} + 2(3)(1)^2 - 3(1) = c \Rightarrow c = 12.$ $\boxed{\text{1 pt.}}$

Therefore, the solution of the initial value problem is $\boxed{\boxed{\dfrac{x^3}{3} + 2xy^2 - 3y^2 = 12}}$

Problem 5. (10 points) Let P denote the population of a colony of tribbles. Suppose that the birth rate β (number of births per week per tribble) is proportional to P and that the death rate δ (number of deaths per week per tribble) equals 0. Suppose the initial population is 2, and after one week the population is 3. What is the population after 2 weeks?

$\dfrac{dP}{dt} = \beta P - \delta P = (kP)P - (0)\,P = kP^2.$ $\boxed{\text{3 pts.}}$

This is a separable d.e: $\dfrac{dP}{dt} = kP^2 \Rightarrow \dfrac{dP}{P^2} = k\,dt \Rightarrow \displaystyle\int P^{-2}\,dP = \int k\,dt \Rightarrow -P^{-1} = kt + c.$

$\boxed{\text{4 pts.}}$

$P(0) = 2 \Rightarrow -2^{-1} = -k(0) + c \Rightarrow c = -\dfrac{1}{2} \Rightarrow -P^{-1} = kt - \dfrac{1}{2} \Rightarrow P^{-1} = \dfrac{1}{2} - kt$ $\boxed{\text{1 pt.}}$

$P(1) = 3 \Rightarrow 3^{-1} = \dfrac{1}{2} - k(1) \Rightarrow\Rightarrow k = \dfrac{1}{6}.$ $\boxed{\text{1 pt.}}$

Therefore, $(P(2))^{-1} = \dfrac{1}{2} - \left(\dfrac{1}{6}\right)(2) = \dfrac{1}{6} \Rightarrow \boxed{\boxed{P(2) = 6 \text{ tribbles}}}.$ $\boxed{\text{1 pt.}}$