MATH.2360 Engineering Differential Equations   Exam # 3 Solutions   Fall 2017

Problem 1. (20 pts.) Solve the following differential equations.

a. (8 pts.) $y'' - 4y' + 8y = 0$

Characteristic equation: $r^2 - 4r + 8 = 0 \Rightarrow r = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)} = \dfrac{4 \pm \sqrt{-16}}{2} = \dfrac{4 \pm 4i}{2} = 2 \pm 2i$

$\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{2x}\cos(2x) + c_2 e^{2x}\sin(2x)}}$ $\boxed{\text{4 pts.}}$

b. (12 pts.) $y^{(4)} + 9y'' = 0$

Characteristic equation: $r^4 + 9r^2 = 0 \Rightarrow r^2\left(r^2+9\right) = 0$. $r^2 + 9 = 0 \Rightarrow r^2 = -9 \Rightarrow r = \pm 3i$. Therefore, the roots of the characterisic equation are $r = 0$ (double root) and $r = \pm 3i = 0 \pm 3i$.
$\boxed{\text{4 pts.}}$ Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{0x}\cos(3x) + c_4 e^{0x}\sin(3x)$, or

$\boxed{\boxed{y = c_1 + c_2 x + c_3\cos(3x) + c_4\sin(3x)}}$ $\boxed{\text{8 pts.}}$

Problem 2. (25 pts.) Solve the following initial value problem:

$$y'' + y' - 6y = 16xe^x,\ y(0) = -2,\ y'(0) = 0.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' + y' - 6y = 0$.
Characteristic equation: $r^2 + r - 6 = 0 \Rightarrow (r+3)(r-2) = 0 \Rightarrow r = -3$ or $r = 2$.
Therefore, $y_c = c_1 e^{-3x} + c_2 e^{2x}$. $\boxed{\text{5 pts.}}$

Step 2. Find y_p.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $16xe^x$ in the given d.e. is a polynomial of degree 1 times an exponential function, we should guess that y_p is a polynomial of degree 1 times an exponential function:
$y_p = (Ax+B)\,e^x$. $\boxed{\text{4 pts.}}$ No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{\text{2 pts.}}$ $y = (Ax+B)\,e^x \Rightarrow y' = Ae^x + (Ax+B)\,e^x = (Ax+A+B)\,e^x \Rightarrow$
$y'' = Ae^x + (Ax+A+B)\,e^x = (Ax+2A+B)\,e^x$. Therefore, the left side of the d.e. is
$y'' + y' - 6y = (Ax+2A+B)\,e^x + [(Ax+A+B)\,e^x] - 6\,[(Ax+B)\,e^x] = (-4Ax+3A-4B)\,e^x$.
We want this to equal the nonhomogeneous term $16xe^x$:
$(-4Ax+3A-4B)\,e^x = 16xe^x \Rightarrow -4A = 16,\ 3A - 4B = 0 \Rightarrow A = -4,\ B = -3$. Thus, $y_p = (-4x-3)\,e^x$. $\boxed{\text{9 pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-3x}$ and $y_2 = e^{2x}$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-3x} & e^{2x} \\ -3e^{-3x} & 2e^{2x} \end{vmatrix} = e^{-3x}\left(2e^{2x}\right) - \left(-3e^{-3x}\right)e^{2x} = 5e^{-x}. \ \boxed{\text{1 pt.}}$$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{2x}\left(16xe^x\right)}{5e^{-x}}\,dx = -\frac{16}{5}\int\left[xe^{4x}\right]\,dx = -\frac{16}{5}\left[\frac{1}{16}(4x-1)\,e^{4x}\right] = -\frac{1}{5}(4x-1)\,e^{4x}$$

using formula 46 from the integral table, with $u = 4x$. $\boxed{\text{4 pts.}}$

$$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-3x}\left(16xe^x\right)}{5e^{-x}}\,dx = \frac{16}{5}\int\left[xe^{-x}\right]\,dx = \frac{16}{5}\left[(-x-1)\,e^{-x}\right] = -\frac{16}{5}(x+1)\,e^{-x}$$

using formula 46 from the integral table, with $u = -x$. $\boxed{\text{4 pts.}}$

Therefore, $y_p = u_1 y_1 + u_2 y_2 = \left[-\dfrac{1}{5}(4x-1)\,e^{4x}\right]e^{-3x} + \left[-\dfrac{16}{5}(x+1)\,e^{-x}\right]e^{2x} = (-4x-3)\,e^x$ $\boxed{\text{5 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1 e^{-3x} + c_2 e^{2x} + (-4x - 3)\, e^x$. $\boxed{\text{3 pts.}}$

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.

$y = y = c_1 e^{-3x} + c_2 e^{2x} + (-4x-3)\, e^x \Rightarrow y' = -3c_1 e^{-3x} + 2c_2 e^{2x} - 4e^x + (-4x-3)\, e^x = -3c_1 e^{-3x} + 2c_2 e^{2x} + (-4x-7)\, e^x$.

$y(0) = -2 \Rightarrow -2 = c_1 e^0 + c_2 e^0 - 3e^0 = c_1 + c_2 - 3 \Rightarrow c_1 + c_2 = 1$

$y'(0) = 0 \Rightarrow 0 = -3c_1 e^0 + 2c_2 e^0 - 7e^0 = -3c_1 + 2c_2 - 7 \Rightarrow -3c_1 + 2c_2 = 7 . c_1 + c_2 = 1,\ -3c_1 + 2c_2 = 7 \Rightarrow c_1 = -1,\ c_2 = 2$ $\boxed{\text{2 pts.}}$ Therefore, $\boxed{\boxed{y = -e^{-3x} + 2e^{2x} + (-4x-3)\, e^x}}$

Problem 3. (20 pts.) Consider a free, undamped mass-spring system with mass $m = 1$ kg and spring constant $k = 25$ N/m. Suppose $x(0) = -1$ and $x'(0) = -5$.

a. Find the position function $x(t)$.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_e(t)$. $\boxed{\text{3 pts.}}$

In this problem, $c = 0$ and $F_e(t) = 0$ so the d.e. becomes $x'' + 25x = 0$. $\boxed{\text{3 pts.}}$

The characteristic equation is $r^2 + 25 = 0$ so $r^2 = -25 \Rightarrow r = \pm 5i \Rightarrow x = c_1 \cos(5t) + c_2 \sin(5t)$ $\boxed{\text{10 pts.}}$

$x(0) = -1 \Rightarrow -1 = c_1 \cos(0) + c_2 \sin(0) = c_1$ $\boxed{\text{1 pt.}}$ so $x = -\cos(5t) + c_2 \sin(5t) \Rightarrow x' = 5\sin(5t) + 5c_2 \cos(5t)$.

$x'(0) = -5 \Rightarrow -5 = 5\sin(0) + 5c_2 \cos(0) = 5c_2 \Rightarrow c_2 = -1$. $\boxed{\text{1 pt.}}$ Therefore, $\boxed{\boxed{x = -\cos(5t) - \sin(5t)}}$

b. Express your solution from part a in the form $x = C\cos(\omega_0 t - \alpha)$.

$x = c_1 \cos(5t) + c_2 \sin(5t)$ where $c_1 = -1$ and $c_2 = -1$

$C = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ $\boxed{\text{1 pt.}}$

Because $c_1 < 0$ we have $\alpha = \pi + \tan^{-1}(c_2/c_1) = \pi + \tan^{-1}((-1)/(-1)) = \pi + \pi/4 = 5\pi/4$. $\boxed{\text{1 pt.}}$

Therefore, $\boxed{\boxed{x = \sqrt{2}\cos(5t - 5\pi/4)}}$

Problem 4. (20 points) Consider an RLC circuit with inductance $L = 1$ henry, resistance $R = 10\Omega$, capacitance $C = 1/9$ farad, and applied voltage $E(t) = 60\sin(3t)$ volts. Find the steady periodic current $I_{sp}(t)$.

The d.e. describing an RLC circuit is $LQ'' + RQ' + \dfrac{Q}{C} = E(t)$. $\boxed{\text{2 pts.}}$

In this problem, the d.e. becomes $Q'' + 10Q' + 9Q = 60\cos(3t)$. $\boxed{\text{1 pt.}}$

The steady periodic solution is the particular solution. $\boxed{\text{3 pts.}}$ Since the nonhomogeneous term $60\sin(3t)$ is a sine, we should guess that Q_p is a combination of a sine and a cosine with the same frequency: $Q_p = A\sin(3t) + B\cos(3t)$. $\boxed{\text{5 pts.}}$ (No part of this guess will duplicate part of Q_c because Q_c is a transient term containing decaying exponential functions.)

$Q = A\sin(3t) + B\cos(3t) \Rightarrow Q' = 3A\cos(3t) - 3B\sin(3t) \Rightarrow$

$Q'' = -9A\sin(3t) - 9B\cos(3t)$. Therefore, the left side of the d.e. is

$Q'' + 10Q' + 9Q = -9A\sin(3t) - 9B\cos(3t) + 10\,[3A\cos(3t) - 3B\sin(3t)] + 9\,[A\sin(3t) + B\cos(3t)]$

$= -30B\sin(3t) + 30A\cos(3t)$.

We want this to equal the nonhomogeneous term $60\sin(3t)$:

$-30B\sin(3t) + 30A\cos(3t) = 60\sin(3t) \Rightarrow -30B = 60,\ 30A = 0 \Rightarrow A = 0$ and $B = -2$. Therefore,

$Q_{\text{sp}} = -2\cos(3t)$. $\boxed{\text{8 pts.}}$ Current is the derivative of Q, so $\boxed{\boxed{I_{\text{sp}} = 6\sin(3t)}}$ $\boxed{\text{1 pt.}}$