92.236 Engineering Differential Equations   Exam # 1 Solutions   Fall 2017

Problem 1. (10 points)

Is $y(x) = x^2 + 1$ is a solution of the d.e. $yy' = x^2 + y$? Why or why not?

Left side of d.e: $y = x^2 + 1 \Rightarrow y' = 2x \Rightarrow yy' = \left(x^2 + 1\right)(2x) = 2x^3 + 2x$. $\boxed{\text{4 pts.}}$

Right side of d.e: $y = x^2 + 1 \Rightarrow x^2 + y = x^2 + \left(x^2 + 1\right) = 2x^2 + 1$. $\boxed{\text{3 pts.}}$

Left side $\neq$ right side, so $\boxed{\boxed{y(x) = x^2 + 1 \textbf{ is not} \text{ a solution of the d.e. } yy' = x^2 + y}}$ $\boxed{\text{3 pts.}}$

Problem 2. (15 points)

A ball is dropped from rest (initial velocity 0) from a height of 45 meters. How long does it take the ball to reach the ground? Use the value 10 m/s^2 for g, the acceleration due to gravity.

Let t denote time in seconds, and let x denote the height of the ball above ground level. This is a constant-acceleration one-dimensional motion problem, so as we showed in class, $x = \frac{1}{2}at^2 + v_0 t + x_0$. $\boxed{\text{5 pts.}}$ Here $a = -g \approx -10$ m/s^2, $v_0 = 0$ m/s, and $x_0 = 45$ m. Therefore, $x = -5t^2 + 45$. $\boxed{\text{5 pts.}}$ Ground level corresponds to $x = 0$. Setting $x = 0$ in the position function gives $0 = -5t^2 + 45 \Rightarrow t^2 = 9 \Rightarrow$ $\boxed{\boxed{t = 3 \text{ seconds}}}$. $\boxed{\text{5 pts.}}$

Problem 3. (25 points)

Solve the following initial value problem:

$$x\frac{dy}{dx} + 2y = 6x, \quad y(1) = 5.$$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone or by constants. $\boxed{\text{5 pts.}}$

First write the equation in standard form:

$x\frac{dy}{dx} + 2y = 6x \Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right)y = 6$ $\boxed{\text{3 pts.}}$

Next, find the integrating factor: $\rho(x) = e^{\int 2/x \, dx} = e^{2\ln(x)} = x^2$. $\boxed{\text{6 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^2\left[\frac{dy}{dx} + \left(\frac{2}{x}\right)y\right] = x^2(6) \Rightarrow x^2\frac{dy}{dx} + 2xy = 6x^2$. $\boxed{\text{2 pts.}}$

Use the Product Rule backwards to rewrite the d.e. as $\frac{d}{dx}\left[x^2 y\right] = 6x^2$. $\boxed{\text{4 pts.}}$

Integrating both sides, we obtain $x^2 y = \int 6x^2 \, dx = 2x^3 + c$. $\boxed{\text{3 pts.}}$

$y(1) = 5 \Rightarrow 1^2(5) = 2(1)^3 + c \Rightarrow c = 3$ $\boxed{\text{2 pts.}}$

Therefore, $x^2 y = 2x^3 + 3$, so $\boxed{\boxed{y = 2x + 3x^{-2}.}}$

Problem 4. (25 points)

Solve the following initial value problem:

$$x\frac{dy}{dx} = \frac{1+2x}{2y}, \quad y(1) = 2.$$

This is a separable d.e. $\boxed{\text{5 pts.}}$

$x\dfrac{dy}{dx} = \dfrac{1+2x}{2y} \Rightarrow 2y\ dy = \left(\dfrac{1+2x}{x}\right)\ dx.$ $\boxed{\text{5 pts.}}$

$\Rightarrow 2y\ dy = \left(\dfrac{1}{x}+2\right)\ dx \Rightarrow \displaystyle\int 2y\ dy = \int \left(\dfrac{1}{x}+2\right)\ dx \Rightarrow y^2 = \ln(x) + 2x + c.$ $\boxed{\text{12 pts.}}$

$y(1) = 2 \Rightarrow 2^2 = \ln(1) + 2(1) + c \Rightarrow c = 2$ $\boxed{\text{3 pts.}}$

$\Rightarrow y^2 = \ln(x) + 2x + 2 \Rightarrow \boxed{\boxed{y = \sqrt{\ln(x) + 2x + 2}}}.$

Problem 5. (15 points)

A tank initially contains 50 liters of water in which 100 grams of salt are dissolved. A salt solution containing 5 grams of salt per liter is pumped into the tank at the rate of 8 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 5 liters per minute.

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in grams). Write down the differential equation $\left(\dfrac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.

$\dfrac{dx}{dt} = \text{rate in - rate out}$ $\boxed{\text{3 pts.}}$

$=$ (flow rate in)(concentration in) - (flow rate out)(concentration out), $\boxed{\text{3 pts.}}$ so

$$\frac{dx}{dt} = \underbrace{\left(8\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(5\frac{\text{gm}}{\text{L}}\right)}_{\boxed{\text{1 pt.}}} - \underbrace{\left(5\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(\frac{x\ \text{gm}}{(50+3t)\ \text{L}}\right)}_{\boxed{\text{5 pts.}}}.$$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 50 + (8-5)t$ liters.)

Initially there are 100 gm of salt in the tank, so $x(0) = 100$ $\boxed{\text{1 pt.}}$.

Therefore, the initial value problem describing this mixing problem is $\boxed{\boxed{\dfrac{dx}{dt} = 40 - \dfrac{5x}{50+3t} \quad \text{with } x(0) = 100.}}$