92.236 Engineering Differential Equations Exam # 1 Solutions Fall 2018

Problem 1. (10 points)

Is $y(x) = \dfrac{1}{x}$ is a solution of the d.e. $y' = \dfrac{2}{x^2} + y^2$? Why or why not?

Left side of d.e: $y = \dfrac{1}{x} = x^{-1} \Rightarrow y' = -x^{-2}$. $\boxed{\text{3 pts.}}$

Right side of d.e: $y = x^{-1} \Rightarrow \dfrac{2}{x^2} + y^2 = \dfrac{2}{x^2} + \left(x^{-1}\right)^2 = 2x^{-2} + x^{-2} = 3x^{-2}$. $\boxed{\text{4 pts.}}$

Left side $\neq$ right side, so $\boxed{\boxed{y(x) = x^2 + 1 \textbf{ is not } \text{a solution of the d.e. } yy' = x^2 + y}}$ $\boxed{\text{3 pts.}}$

Problem 2. (15 points)

A car travels 20 meters in 5 seconds after the driver applies the brakes. What was the initial velocity of the car? Assume the car's deceleration is constant.

Let t denote time in seconds, and let x denote the position of the car, measured in meters from the point at which the brakes were applied. This is a constant-acceleration one-dimensional motion problem, so as we showed in class, $x = \dfrac{1}{2}at^2 + v_0 t + x_0$ and $v = at + v_0$. $\boxed{\text{5 pts.}}$ At $t = 5$ s we have $x = 20$ m and $v = 0$ m/s: $20 = \dfrac{1}{2}a(5)^2 + v_0(5)$, $0 = a(5) + v_0$. $\boxed{\text{6 pts.}}$ The second equation implies $a = -v_0/5$. Substituting $-v_0/5$ for a in the first equation gives $20 = \dfrac{1}{2}\left(-\dfrac{v_0}{5}\right)(5)^2 + v_0(5) = \dfrac{5v_0}{2} \Rightarrow$ $\boxed{\boxed{v_0 = 8 \text{ m/s}}}$. $\boxed{\text{4 pts.}}$

Problem 3. (25 points)

Solve the following initial value problem:

$$x^2 \frac{dy}{dx} = xy + 4, \quad y(1) = 0.$$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone. $\boxed{\text{5 pts.}}$ First write the equation in standard form:

$x^2 \dfrac{dy}{dx} = xy + 4 \Rightarrow \dfrac{dy}{dx} - \left(\dfrac{1}{x}\right) y = \dfrac{4}{x^2}$ $\boxed{\text{3 pts.}}$

Next, find the integrating factor: $\rho(x) = e^{\int -1/x \, dx} = e^{-\ln(x)} = x^{-1}$. $\boxed{\text{6 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^{-1}\left[\dfrac{dy}{dx} - \left(\dfrac{1}{x}\right) y\right] = x^{-1}\left(\dfrac{4}{x^2}\right) \Rightarrow x^{-1}\dfrac{dy}{dx} - x^{-2}y = 4x^{-3}$. $\boxed{\text{2 pts.}}$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\left[x^{-1}y\right] = 4x^{-3}$. $\boxed{\text{4 pts.}}$

Integrating both sides, we obtain $x^{-1}y = \displaystyle\int 4x^{-3} \, dx = -2x^{-2} + c$. $\boxed{\text{3 pts.}}$

$y(1) = 0 \Rightarrow 1^{-1}(0) = -2(1)^{-2} + c \Rightarrow c = 2$ $\boxed{\text{2 pts.}}$

Therefore, $x^{-1}y = -2x^{-2} + 2$, so $\boxed{\boxed{y = -2x^{-1} + 2x.}}$

Problem 4. (25 points)

Solve the following initial value problem:

$$x\frac{dy}{dx} = y^2 - xy^2, \quad y(1) = 1.$$

This is a separable d.e. $\boxed{\text{5 pts.}}$

$x\dfrac{dy}{dx} = y^2 - xy^2 \Rightarrow x\dfrac{dy}{dx} = y^2(1-x) \Rightarrow \dfrac{dy}{y^2} = \left(\dfrac{1-x}{x}\right)\, dx.$ $\boxed{\text{5 pts.}}$

$\Rightarrow \displaystyle\int y^{-2}\, dy = \int \left(\frac{1}{x} - 1\right)\, dx \Rightarrow -y^{-1} = \ln(x) - x + c.$ $\boxed{\text{12 pts.}}$

$y(1) = 1 \Rightarrow -1^{-1} = \ln(1) - 1 + c \Rightarrow c = 0$ $\boxed{\text{3 pts.}}$

$\Rightarrow -y^{-1} = \ln(x) - x \Rightarrow \boxed{\boxed{y = \dfrac{1}{x - \ln(x)}}}.$

Problem 5. (15 points)

A tank initially contains 80 liters of water in which 40 grams of salt are dissolved. A salt solution containing 4 grams of salt per liter is pumped into the tank at the rate of 5 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 7 liters per minute.

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in grams). Write down the differential equation $\left(\dfrac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.

$\dfrac{dx}{dt} = \text{rate in - rate out}$ $\boxed{\text{3 pts.}}$

$=$ (flow rate in)(concentration in) - (flow rate out)(concentration out), $\boxed{\text{3 pts.}}$ so

$$\frac{dx}{dt} = \underbrace{\left(5\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(4\frac{\text{gm}}{\text{L}}\right)}_{\boxed{\text{1 pt.}}} - \underbrace{\left(7\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(\frac{x\text{ gm}}{(80-2t)\text{ L}}\right)}_{\boxed{\text{5 pts.}}}.$$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 80 + (5-7)t$ liters.)

Initially there are 40 gm of salt in the tank, so $x(0) = 40$ $\boxed{\text{1 pt.}}$.

Therefore, the initial value problem describing this mixing problem is $\boxed{\boxed{\dfrac{dx}{dt} = 20 - \dfrac{7x}{80-2t} \quad \text{with } x(0) = 40.}}$