92.236 Engineering Differential Equations Exam # 1 Solutions Fall 2018

Problem 1. (10 points)

Is $y(x) = \frac{1}{x}$ is a solution of the d.e. $y' = -\frac{1}{x^2} + y^2$? Why or why not?

Left side of d.e: $y = \frac{1}{x} = x^{-1} \Rightarrow y' = -x^{-2}$. $\boxed{\text{3 pts.}}$

Right side of d.e: $y = x^{-1} \Rightarrow -\frac{1}{x^2} + y^2 = -\frac{1}{x^2} + \left(x^{-1}\right)^2 = -x^{-2} + x^{-2} = 0$. $\boxed{\text{4 pts.}}$

Left side $\neq$ right side, so $\boxed{\boxed{y(x) = x^2 + 1 \textbf{ is not} \text{ a solution of the d.e. } yy' = x^2 + y}}$ $\boxed{\text{3 pts.}}$

Problem 2. (15 points)

A car is traveling at a speed of 10 m/s when the brakes are applied. The car continues traveling for 5 seconds before coming to a stop. How far (in meters) does the car travel during those 5 seconds? Assume the car's deceleration is constant.

Let t denote time in seconds, and let x denote the position of the car, measured in meters from the point at which the brakes were applied. This is a constant-acceleration one-dimensional motion problem, so as we showed in class, $x = \frac{1}{2}at^2 + v_0 t + x_0 = \frac{1}{2}at^2 + 10t$ and $v = at + v_0 = at + 10$. $\boxed{\text{5 pts.}}$ At $t = 5$ s we have $v = 0$ m/s: $0 = a(5) + 10 \Rightarrow a = -2$. $\boxed{\text{5 pts.}}$ Therefore, $x = -t^2 + 10t$ so $\boxed{\boxed{x(5) = 25 \text{ m}}}$. $\boxed{\text{5 pts.}}$

Problem 3. (25 points)

Solve the following initial value problem:

$$x\frac{dy}{dx} = y^2 - xy^2, \quad y(1) = 1.$$

This is a separable d.e. $\boxed{\text{5 pts.}}$

$x\frac{dy}{dx} = y^2 - xy^2 \Rightarrow x\frac{dy}{dx} = y^2(1-x) \Rightarrow \frac{dy}{y^2} = \left(\frac{1-x}{x}\right)\, dx$. $\boxed{\text{5 pts.}}$

$\Rightarrow \int y^{-2}\, dy = \int \left(x^{-1} - 1\right)\, dx \Rightarrow -y^{-1} = \ln(x) - x + c$. $\boxed{\text{12 pts.}}$

$y(1) = 1 \Rightarrow -1^{-1} = \ln(1) - 1 + c \Rightarrow c = 0$ $\boxed{\text{3 pts.}}$

$\Rightarrow -y^{-1} = \ln(x) - x \Rightarrow \boxed{\boxed{y = \frac{1}{x - \ln(x)}}}$.

Problem 4. (25 points)

Solve the following initial value problem:

$$x^2\frac{dy}{dx} = 2xy + 9, \quad y(1) = 3.$$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone. $\boxed{\text{5 pts.}}$ First write the equation in standard form:

$$x^2\frac{dy}{dx} = 2xy + 9 \Rightarrow \frac{dy}{dx} - \left(\frac{2}{x}\right)y = \frac{9}{x^2} \quad \boxed{\text{3 pts.}}$$

Next, find the integrating factor: $\rho(x) = e^{\int -2/x\ dx} = e^{-2\ln(x)} = x^{-2}$. $\boxed{\text{6 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$$x^{-2}\left[\frac{dy}{dx} - \left(\frac{2}{x}\right)y\right] = x^{-2}\left(\frac{9}{x^2}\right) \Rightarrow x^{-2}\frac{dy}{dx} - 2x^{-3}y = 9x^{-4}. \quad \boxed{\text{2 pts.}}$$

Use the Product Rule backwards to rewrite the d.e. as $\frac{d}{dx}\left[x^{-2}y\right] = 9x^{-4}$. $\boxed{\text{4 pts.}}$

Integrating both sides, we obtain $x^{-2}y = \int 9x^{-4}\ dx = -3x^{-3} + c$. $\boxed{\text{3 pts.}}$

$y(1) = 3 \Rightarrow 1^{-1}(3) = -3(1)^{-3} + c \Rightarrow c = 6$ $\boxed{\text{2 pts.}}$

Therefore, $x^{-2}y = -3x^{-3} + 6$, so $\boxed{\boxed{y = -3x^{-1} + 6x^2.}}$

Problem 5. (15 points)

A tank initially contains 90 liters of water in which 30 grams of salt are dissolved. A salt solution containing 5 grams of salt per liter is pumped into the tank at the rate of 7 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 4 liters per minute.

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in grams). Write down the differential equation $\left(\frac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.

$\frac{dx}{dt} = \text{rate in - rate out}$ $\boxed{\text{3 pts.}}$

$=$ (flow rate in)(concentration in) - (flow rate out)(concentration out), $\boxed{\text{3 pts.}}$ so

$$\frac{dx}{dt} = \underbrace{\left(7\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(5\frac{\text{gm}}{\text{L}}\right)}_{\boxed{\text{1 pt.}}} - \underbrace{\left(4\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(\frac{x\text{ gm}}{(90+3t)\text{ L}}\right)}_{\boxed{\text{5 pts.}}}.$$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 90 + (7-4)t$ liters.)

Initially there are 30 gm of salt in the tank, so $x(0) = 30$ $\boxed{\text{1 pt.}}$.

Therefore, the initial value problem describing this mixing problem is $\boxed{\boxed{\frac{dx}{dt} = 35 - \frac{4x}{90+3t} \quad \text{with } x(0) = 30.}}$