**92.236 Engineering Differential Equations  Exam # 2 Solutions  Fall 2018**

**Problem 1. 20 points** Consider the autonomous differential equation $\frac{dx}{dt} = x^2 - x - 2$.

a. Find all critical points (equilibrium solutions) of this d.e.

$x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow$
$\boxed{\text{the equilibrium solutions are } x = -1 \text{ and } x = 2}$ $\boxed{\text{3 pts.}}$

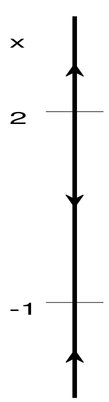
b. Draw the phase line (phase diagram) for this d.e. $\boxed{\text{8 pts.}}$

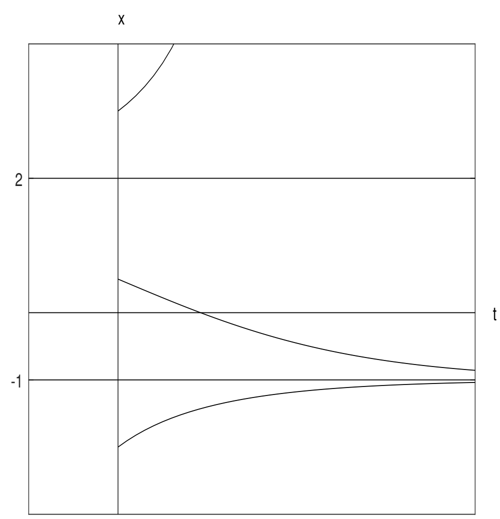
The two equilibrium solutions divide the phase line into 3 intervals: $x > 2$, $-1 < x < 2$, and $x < -1$.

$\left.\frac{dx}{dt}\right|_{x=3} = (3+1)(3-2) > 0$, so the direction arrow points up for $x > 2$.

$\left.\frac{dx}{dt}\right|_{x=0} = (0+1)(0-2) < 0$, so the direction arrow points down for $-1 < x < 2$.

$\left.\frac{dx}{dt}\right|_{x=-2} = (-2+1)(-2-2) > 0$, so the direction arrow points up for $x < -1$.




c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{-1 \text{ is stable and } 2 \text{ is unstable}}$. $\boxed{\text{2 pts.}}$

d. If $x(0) = -2$, what value will $x(t)$ approach as $t$ increases?

Since $-2$ lies in the interval $x < -1$, we can see from the phase line that $\boxed{x(t) \to -1}$ as $t$ increases. $\boxed{\text{3 pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. $\boxed{\text{4 pts.}}$

See the figure above.

**Problem 2. (20 points)** Solve the following differential equations:

a. $y'' + 8y' + 15y = 0$

Characteristic equation: $r^2 + 8r + 15 = 0 \Rightarrow (r+5)(r+3) = 0 \Rightarrow$
$r = -5$ or $r = -3$. $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-5x} + c_2 e^{-3x}}}$ $\boxed{\text{6 pts.}}$

b. $y'' + 4y' + 4y = 0$

Characteristic equation: $r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow$
$r = -2$ repeated root. $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-2x} + c_2 x e^{-2x}}}$ $\boxed{\text{6 pts.}}$

**Problem 3. (20 points)** Solve the following initial value problem.

$$2xy^2 \frac{dy}{dx} + x^3 - 2y^3 = 0, \quad y(1) = 3.$$

$2xy^2 \frac{dy}{dx} + x^3 - 2y^3 = 0 \Rightarrow \frac{dy}{dx} = \frac{2y^3 - x^3}{2xy^2}$. $dy/dx$ equals a rational function, and every term has the same degree (3). Therefore, this d.e. is homogeneous. $\boxed{\text{4 pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\frac{dy}{dx}$ by $v + x\frac{dv}{dx}$ and we replace $y$ by $xv$:

$$\frac{dy}{dx} = \frac{2y^3 - x^3}{2xy^2} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{2(xv)^3 - x^3}{2x(xv)^2}}_{\boxed{\text{4 pts.}}} = \frac{x^3(2v^3 - 1)}{2x^3v^2} = \frac{2v^3 - 1}{2v^2} \Rightarrow \underbrace{x\frac{dv}{dx} = \frac{2v^3 - 1}{2v^2} - v = \frac{-1}{2v^2}}_{\boxed{\text{3 pts.}}}$$

$$\underbrace{\Rightarrow 2v^2\, dv = \frac{-1}{x}\, dx}_{\boxed{\text{2 pts.}}} \Rightarrow \int 2v^2\, dv = \int \frac{-1}{x}\, dx \Rightarrow \underbrace{\frac{2v^3}{3} = -\ln(x) + c}_{\boxed{\text{3 pts.}}} \Rightarrow \underbrace{\frac{2}{3}\left(\frac{y}{x}\right)^3 = -\ln(x) + c}_{\boxed{\text{2 pts.}}}$$

The initial condition $y(1) = 3 \Rightarrow \frac{2}{3}\left(\frac{3}{1}\right)^3 = -\ln(1) + c \Rightarrow c = 18$ $\boxed{\text{2 pts.}}$.

Therefore, $\frac{2}{3}\left(\frac{y}{x}\right)^3 = -\ln(x) + 18 \Rightarrow \boxed{\boxed{y = x\left[\frac{3}{2}(-\ln(x) + 18)\right]^{1/3}}}$.

**Problem 4. (20 points)** Solve the following initial value problem.

$$2xy + 4x^3 + \left[x^2 + 6y^2\right]\frac{dy}{dx} = 0, \quad y(-1) = 1.$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{2xy + 4x^3}_{M} + \underbrace{\left[x^2 + 6y^2\right]}_{N}\frac{dy}{dx} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[2xy + 4x^3\right] = 2x.$ $\boxed{\text{1 pt.}}$ $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[x^2 + 6y^2\right] = 2x.$ $\boxed{\text{1 pt.}}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{3 pts.}}$ Therefore, the solution of the d.e. is $f(x, y) = c$, where the function $f$ satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 2xy + 4x^3$ and $\dfrac{\partial f}{\partial y} = N = x^2 + 6y^2$.

$\dfrac{\partial f}{\partial x} = 2xy + 4x^3 \Rightarrow f = \displaystyle\int \left(2xy + 4x^3\right)\, \partial x = x^2y + x^4 + g(y)$ $\boxed{\text{6 pts.}}$

$\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[x^2y + x^4 + g(y)\right] = x^2 + g'(y)$

But $\dfrac{\partial f}{\partial y} = N = x^2 + 6y^2 \Rightarrow x^2 + g'(y) = x^2 + 6y^2 \Rightarrow g'(y) = 6y^2 \Rightarrow g(y) = 2y^3 \Rightarrow$

$f = x^2y + x^4 + 2y^3$ $\boxed{\text{6 pts.}}$

Therefore, the solution of the d.e. is $x^2y + x^4 + 2y^3 = c$ $\boxed{\text{2 pts.}}$

$y(-1) = 1 \Rightarrow (-1)^2(1) + (-1)^4 + 2(1)^3 = c \Rightarrow c = 4.$ $\boxed{\text{1 pt.}}$

Therefore, the solution of the initial value problem is $\boxed{\boxed{x^2y + x^4 + 2y^3 = 4}}$

**Problem 5. (10 points)** Let $P$ denote the population of a colony of tribbles. Suppose that $\beta$ (the number of births per week per tribble) is proportional to $P$ and that $\delta$ (the number of deaths per week per tribble) equals 0. Suppose the initial population is 2 and the population after 5 weeks is 4. When will the population reach 20?

$\dfrac{dP}{dt} = \beta P - \delta P = (kP))P - (0)\,P = kP^2.$ $\boxed{\text{3 pts.}}$

This is a separable d.e: $\dfrac{dP}{dt} = kP^2 \Rightarrow \dfrac{dP}{P^2} = k\ dt \Rightarrow \displaystyle\int P^{-2}\ dP = \int k\ dt \Rightarrow -P^{-1} = kt + c.$

$\boxed{\text{4 pts.}}$

$P(0) = 2 \Rightarrow -(2)^{-1} = k(0) + c \Rightarrow c = -0.5 \Rightarrow -P^{-1} = kt - 0.5 \Rightarrow P^{-1} = 0.5 - kt$ $\boxed{\text{1 pt.}}$

$P(5) = 4 \Rightarrow (4)^{-1} = 0.5 - k(5) \Rightarrow -0.25 = -5k \Rightarrow k = 0.05 \Rightarrow P^{-1} = 0.5 - 0.05t.$ $\boxed{\text{1 pt.}}$

Therefore, $P(t) = 20 \Rightarrow (20)^{-1} = 0.5 - 0.05t \Rightarrow \boxed{\boxed{t = 9 \text{ weeks}}}$. $\boxed{\text{1 pt.}}$