Problem 1. (20 pts.) Solve the following differential equations.

a. (8 pts.) $y'' + 2y' + 10y = 0$

Characteristic equation: $r^2 + 2r + 10 = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(10)}}{2(1)} = \dfrac{-2 \pm \sqrt{-36}}{2} = \dfrac{-2 \pm 6i}{2} = -1 \pm 3i$

$\boxed{\text{4 pts.}}$

Therefore, $\boxed{\boxed{y = c_1 e^{-x}\cos(3x) + c_2 e^{-x}\sin(3x)}}$ $\boxed{\text{4 pts.}}$

b. (12 pts.) $y^{(4)} + 4y'' = 0$

Characteristic equation: $r^4 + 4r^2 = 0 \Rightarrow r^2(r^2+4) = 0 \Rightarrow r = \pm 2i$ or $r = 0$ (double root).
$\boxed{\text{4 pts.}}$ Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{0x}\cos(2x) + c_4 e^{0x}\sin(2x)$, or

$\boxed{\boxed{y = c_1 + c_2 x + c_3\cos(2x) + c_4\sin(2x)}}$ $\boxed{\text{8 pts.}}$

Problem 2. (25 pts.) Solve the following initial value problem:

$$y'' - 4y = 8 + 3e^{-x},\ y(0) = 2,\ y'(0) = 3.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' - 4y = 0$.
Characteristic equation: $r^2 - 4 = 0 \Rightarrow (r+2)(r-2) = 0 \Rightarrow r = -2$ or $r = 2$.
Therefore, $y_c = c_1 e^{-2x} + c_2 e^{2x}$. $\boxed{\text{5 pts.}}$

Step 2. Find y_p.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $8 + 3e^{-x}$ in the given d.e. is a polynomial of degree 0 plus an exponential function, we should guess that y_p is a polynomial of degree 0 plus an exponential function:
$y_p = A + Be^{-x}$. $\boxed{\text{4 pts.}}$ No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{\text{2 pts.}}$
$y = A + Be^{-x} \Rightarrow y' = -Be{-x} \Rightarrow y'' = Be^{-x}$. Therefore, the left side of the d.e. is
$y'' - 4y = Be^{-x} - 4\left[A + Be^{-x}\right] = -4A - 3Be^{-x}$. We want this to equal the nonhomogeneous term $8 + 3e^{-x}$:
$-4A - 3Be^{-x} = 8 + 3e^{-x} \Rightarrow -4A = 8,\ -3B = 3 \Rightarrow A = -2,\ B = -1$. Thus, $y_p = -2 - e^{-x}$.
$\boxed{\text{9 pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-2x}$ and $y_2 = e^{2x}$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = e^{-2x}\left(2e^{2x}\right) - \left(-2e^{-2x}\right)e^{2x} = 4. \ \boxed{\text{1 pt.}}$$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{2x}\left(8 + 3e^{-x}\right)}{4}\,dx = -\frac{1}{4}\int 8e^{2x} + 3e^{x}\,dx =$$
$$-\frac{1}{4}\left(4e^{2x} + 3e^{x}\right) = -e^{2x} - \frac{3}{4}e^{x} \ \boxed{\text{4 pts.}}$$

$$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-2x}\left(8 + 3e^{-x}\right)}{4}\,dx = \frac{1}{4}\int \left[8e^{-2x} + 3e^{-3x}\right]\,dx = \frac{1}{4}\left(-4e^{-2x} - e^{-3x}\right) = -e^{-2x} - \frac{1}{4}e^{-3x}.$$

$\boxed{\text{4 pts.}}$

Therefore, $y_p = u_1y_1 + u_2y_2 = \left[-e^{2x} - \frac{3}{4}e^{x}\right] e^{-2x} + \left[-e^{-2x} - \frac{1}{4}e^{-3x}\right] e^{2x} =$
$-1 - \frac{3}{4}e^{-x} - 1 - \frac{1}{4}e^{-x} = -2 - e^{-x}$ $\boxed{\text{5 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1e^{-2x} + c_2e^{2x} - 2 - e^{-x}$. $\boxed{\text{3 pts.}}$

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.
$y = c_1e^{-2x} + c_2e^{2x} - 2 - e^{-x} \Rightarrow y' = -2c_1e^{-2x} + 2c_2e^{2x} + e^{-x}$.
$y(0) = 2 \Rightarrow 2 = c_1e^0 + c_2e^0 - 2 - e^0 = c_1 + c_2 - 3 \Rightarrow c_1 + c_2 = 5$
$y'(0) = 3 \Rightarrow 3 = -2c_1e^0 + 2c_2e^0 + e^0 = -2c_1 + 2c_2 + 1 \Rightarrow -2c_1 + 2c_2 = 2$
$c_1 + c_2 = 5, \ -2c_1 + 2c_2 = 2 \Rightarrow c_1 = 2, \ c_2 = 3$ $\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = 2e^{-2x} + 3e^{2x} - 2 - e^{-x}}}$

Problem 3. (20 points) Consider a forced, damped mass-spring system with mass $m = 1$ kg, damping constant $c = 10$ N·s/m, spring constant $k = 9$ N/m, and external force $F_{\text{ext}} = 60\cos(3t)$ N. Find the steady-state (steady periodic) solution x_{sp}.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_{\text{e}}(t)$. $\boxed{\text{2 pts.}}$
In this problem, the d.e. becomes $x'' + 10x' + 9x = 60\cos(3t)$. $\boxed{\text{2 pts.}}$

The steady periodic solution is the particular solution x_p. $\boxed{\text{4 pts.}}$ Since the nonhomogeneous term $60\cos(3t)$ is a cosine, we should guess that x_p is a combination of a cosine and a sine with the same frequency: $x_p = A\cos(3t) + B\sin(3t)$. $\boxed{\text{5 pts.}}$ (No part of this guess will duplicate part of x_c because x_c is a transient term containing decaying exponential functions.)
$x = A\cos(3t) + B\sin(3t) \Rightarrow x' = -3A\sin(3t) + 3B\cos(3t) \Rightarrow x'' = -9A\cos(3t) - 9B\sin(3t)$. Therefore, the left side of the d.e. is
$x'' + 10x' + 9x = -9A\cos(3t) - 9B\sin(3t) + 10\left[-3A\sin(3t) + 3B\cos(3t))\right] + 9\left[A\cos(3t) + B\sin(3t)\right]$
$= 30B\cos(3t) - 30A\sin(3t)$.
We want this to equal the nonhomogeneous term $60\cos(3t)$:
$30B\cos(3t) - 30A\sin(3t) = 60\cos(3t) \Rightarrow 30B = 60, \ -30A = 0 \Rightarrow A = 0$ and $B = 2$. Therefore,
$\boxed{\boxed{x_{\text{sp}} = 2\sin(3t)}}$. $\boxed{\text{7 pts.}}$

Problem 4. (20 points) Solve the system $\begin{cases} x' &= 2y \\ y' &= 3x + y \end{cases}$

Note: $x' = dx/dt$ and $y' = dy/dt$. t is the independent variable.

Take the derivative of both sides of the first d.e. in the system: $x' = 2y \Rightarrow x'' = 2y'$. The second d.e. in the system is $y' = 3x + y$. Therefore, $x'' = 2(3x + y) = 6x + 2y$. From the first d.e. in the system, $2y = x'$, so we have $x'' = 6x + x'$ $\boxed{\text{8 pts.}}$

$x'' = 6x + x' \Rightarrow x'' - x' - 6x = 0$.
Characteristic equation: $r^2 - r - 6 = 0 \Rightarrow (r+2)(r-3) = 0 \Rightarrow r = -2$ or $r = 3 \Rightarrow x = c_1e^{-2t} + c_2e^{3t}$.
$\boxed{\text{8 pts.}}$

The first d.e. in the given system says $y = x'/2$, so $y = \left(-2c_1e^{-2t} + 3c_2e^{3t}\right)/2$. Therefore, the solution of the given system is $\boxed{\boxed{x = c_1e^{-2t} + c_2e^{3t}, \ y = -c_1e^{-2t} + \frac{3}{2}c_2e^{3t}}}$ $\boxed{\text{4 pts.}}$