92.236 Engineering Differential Equations Exam # 2 Solutions Fall 2018

Problem 1. 20 points Consider the autonomous differential equation $\frac{dx}{dt} = x^2 + 2x - 3$.

a. Find all critical points (equilibrium solutions) of this d.e.

$x^2 + 2x - 3 = 0 \Rightarrow (x+3)(x-1) = 0 \Rightarrow$
$\boxed{\boxed{\text{the equilibrium solutions are } x = -3 \text{ and } x = 1}}$ $\boxed{\text{3 pts.}}$

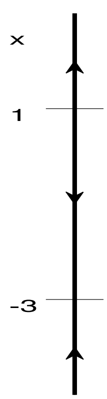
b. Draw the phase line (phase diagram) for this d.e. $\boxed{\text{8 pts.}}$

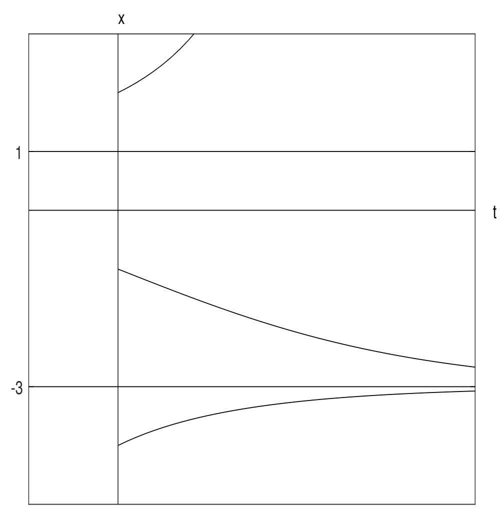
The two equilibrium solutions divide the phase line into 3 intervals: $x > 1$, $-3 < x < 1$, and $x < -3$.

$\left.\frac{dx}{dt}\right|_{x=2} = (2+3)(2-1) > 0$, so the direction arrow points up for $x > 1$.

$\left.\frac{dx}{dt}\right|_{x=0} = (0+3)(0-1) < 0$, so the direction arrow points down for $-3 < x < 1$.

$\left.\frac{dx}{dt}\right|_{x=-4} = (-4+3)(-4-1) > 0$, so the direction arrow points up for $x < -3$.





c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\boxed{-3 \text{ is stable and } 1 \text{ is unstable}}}$. $\boxed{\text{2 pts.}}$

d. If $x(0) = -2$, what value will $x(t)$ approach as t increases?

Since -2 lies in the interval $-3 < x < 1$, we can see from the phase line that $\boxed{\boxed{x(t) \to -3}}$ as t increases. $\boxed{\text{3 pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. 4 pts.

See the figure above.

Problem 2. (20 points) Solve the following differential equations:

a. $y'' + 6y' + 5y = 0$

Characteristic equation: $r^2 + 6r + 5 = 0 \Rightarrow (r+5)(r+1) = 0 \Rightarrow$
$r = -5$ or $r = -1$. 4 pts. Therefore, $\boxed{\boxed{y = c_1 e^{-5x} + c_2 e^{-x}}}$ 6 pts.

b. $y'' + 8y' + 16y = 0$

Characteristic equation: $r^2 + 8r + 16 = 0 \Rightarrow (r+4)^2 = 0 \Rightarrow$
$r = -4$ repeated root. 4 pts. Therefore, $\boxed{\boxed{y = c_1 e^{-4x} + c_2 x e^{-4x}}}$ 6 pts.

Problem 3. (20 points) Solve the following initial value problem.

$$3xy^2 \frac{dy}{dx} + y^3 + 2x = 0, \quad y(1) = 2.$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{\left[3xy^2\right]}_{N} \frac{dy}{dx} + \underbrace{y^3 + 2x}_{M} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[y^3 + 2x\right] = 3y^2.$ 1 pt. $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[3xy^2\right] = 3y^2.$ 1 pt.

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. 3 pts. Therefore, the solution of the d.e. is $f(x, y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = y^3 + 2x$ and $\dfrac{\partial f}{\partial y} = N = 3xy^2$.

$\dfrac{\partial f}{\partial x} = y^3 + 2x \Rightarrow f = \displaystyle\int \left(y^3 + 2x\right)\, \partial x = xy^3 + x^2 + g(y)$ 6 pts.

$\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[xy^3 + x^2 + g(y)\right] = 3xy^2 + g'(y)$

But $\dfrac{\partial f}{\partial y} = N = 3xy^2 \Rightarrow 3xy^2 + g'(y) = 3xy^2 \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0 \Rightarrow$
$f = xy^3 + x^2$ 6 pts.

Therefore, the solution of the d.e. is $xy^3 + x^2 = c$ 2 pts.
$y(1) = 2 \Rightarrow (1)(2)^3 + (1)^2 = c \Rightarrow c = 9.$ 1 pt.

Therefore, the solution of the initial value problem is $xy^3 + x^2 = 9$, or $\boxed{y = \left(\frac{9 - x^2}{x}\right)^{1/3}}$

Problem 4. (20 points) Solve the following initial value problem.

$$3xy^2\frac{dy}{dx} + 2x^3 - 3y^3 = 0, \quad y(1) = 2.$$

$3xy^2\frac{dy}{dx} + 2x^3 - 3y^3 = 0 \Rightarrow \frac{dy}{dx} = \frac{3y^3 - 2x^3}{3xy^2}$. dy/dx equals a rational function, and every term has the same degree (3). Therefore, this d.e. is homogeneous. $\boxed{\text{4 pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\frac{dy}{dx}$ by $v + x\frac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{3y^3 - 2x^3}{3xy^2} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{3(xv)^3 - 2x^3}{3x(xv)^2}}_{\boxed{\text{4 pts.}}} = \frac{x^3(3v^3 - 2)}{3x^3v^2} = \frac{3v^3 - 2}{3v^2} \Rightarrow \underbrace{x\frac{dv}{dx} = \frac{3v^3 - 2}{3v^2} - v = \frac{-2}{3v^2}}_{\boxed{\text{3 pts.}}}$$

$$\underbrace{\Rightarrow 3v^2\, dv = \frac{-2}{x}\, dx}_{\boxed{\text{2 pts.}}} \Rightarrow \int 3v^2\, dv = \int \frac{-2}{x}\, dx \Rightarrow \underbrace{v^3 = -2\ln(x) + c}_{\boxed{\text{3 pts.}}} \Rightarrow \underbrace{\left(\frac{y}{x}\right)^3 = -2\ln(x) + c}_{\boxed{\text{2 pts.}}}$$

The initial condition $y(1) = 2 \Rightarrow \left(\frac{2}{1}\right)^3 = -2\ln(1) + c \Rightarrow c = 8$ $\boxed{\text{2 pts.}}$.

Therefore, $\left(\frac{y}{x}\right)^3 = -2\ln(x) + 8 \Rightarrow \boxed{y = x\left[8 - 2\ln(x)\right]^{1/3}}$.

Problem 5. (10 points) Let P denote the population of a colony of tribbles. Suppose that β (the number of births per week per tribble) is proportional to P and that δ (the number of deaths per week per tribble) equals 0. Suppose the initial population is 5 and the population after 10 weeks is 10. When will the population reach 50?

$\frac{dP}{dt} = \beta P - \delta P = (kP))P - (0)\,P = kP^2$. $\boxed{\text{3 pts.}}$

This is a separable d.e: $\frac{dP}{dt} = kP^2 \Rightarrow \frac{dP}{P^2} = k\, dt \Rightarrow \int P^{-2}\, dP = \int k\, dt \Rightarrow -P^{-1} = kt + c$. $\boxed{\text{4 pts.}}$

$P(0) = 5 \Rightarrow -(5)^{-1} = k(0) + c \Rightarrow c = -0.2 \Rightarrow -P^{-1} = kt - 0.2 \Rightarrow P^{-1} = 0.2 - kt$ $\boxed{\text{1 pt.}}$

$P(10) = 10 \Rightarrow (10)^{-1} = 0.2 - k(10) \Rightarrow -0.1 = -10k \Rightarrow k = 0.01 \Rightarrow P^{-1} = 0.2 - 0.01t$. $\boxed{\text{1 pt.}}$

Therefore, $P(t) = 50 \Rightarrow (50)^{-1} = 0.2 - 0.01t \Rightarrow \boxed{t = 18 \text{ weeks}}$. $\boxed{\text{1 pt.}}$