MATH.2360 Engineering Differential Equations Exam # 3 Solutions Fall 2018

Problem 1. (20 pts.) Solve the following differential equations.

a. (8 pts.) $y'' - 4y' + 20y = 0$

Characteristic equation: $r^2 - 4r + 20 = 0 \Rightarrow r = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(20)}}{2(1)} = \dfrac{4 \pm \sqrt{-64}}{2} = \dfrac{4 \pm 8i}{2} = 2 \pm 4i$

$\boxed{4 \text{ pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{2x}\cos(4x) + c_2 e^{2x}\sin(4x)}}$ $\boxed{4 \text{ pts.}}$

b. (12 pts.) $y^{(3)} + 2y'' + y' = 0.$

Char. eqn.: $r^3 + 2r^2 + r = 0 \Rightarrow r\left(r^2 + 2r + 1\right) = 0 \Rightarrow r(r+1)^2 = 0$ $r = 0$ or $r = -1$ (double root).

$\boxed{4 \text{ pts.}}$ Therefore, $y = c_1 e^{0x} + c_2 e^{-x} + c_3 x e^{-x}$, or $\boxed{\boxed{y = c_1 + c_2 e^{-x} + c_3 x e^{-x}}}$ $\boxed{8 \text{ pts.}}$

Problem 2. (25 pts.) Solve the following initial value problem:

$$y'' + 4y = 8x + 8e^{2x},\ y(0) = 1,\ y'(0) = 2.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' + 4y = 0$.
Characteristic equation: $r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \sqrt{-4} = \pm 2i$.
Therefore, $y_c = c_1\cos(2x) + c_2\sin(2x)$. $\boxed{5 \text{ pts.}}$

Step 2. Find y_p.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $8x + 8e^{2x}$ in the given d.e. is a polynomial of degree 1 plus an exponential function, we should guess that y_p is a polynomial of degree 1 plus an exponential function:
$y_p = Ax + B + Ce^{2x}$. $\boxed{4 \text{ pts.}}$ No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{2 \text{ pts.}}$
$y = Ax + B + Ce^{2x} \Rightarrow y' = A + 2C2x \Rightarrow y'' = 4Ce^{2x}$. Therefore, the left side of the d.e. is $y'' + 4y = 4Ce^{2x} + 4\left[Ax + B + Ce^{2x}\right] = 4Ax + 4B + 8Ce^{2x}$. We want this to equal the nonhomogeneous term $8x + 8e^{2x}$:
$4Ax + 4B + 8Ce^{2x} = 8x + 8e^{2x} \Rightarrow 4A = 8,\ 4B = 0,\ 8C = 8 \Rightarrow A = 2,\ B = 0,\ C = 1$. Thus, $y_p = 2x + e^{2x}$. $\boxed{9 \text{ pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. $\boxed{1 \text{ pt.}}$ The Wronskian is given by

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = \cos(2x)\left(2\cos(2x)\right) - \left(-2\sin(2x)\right)\sin(2x) =$
$2\cos^2(2x) + 2\sin^2(2x) = 2.$ $\boxed{1 \text{ pt.}}$

$u_1 = \displaystyle\int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{\sin(2x)\left(8x + 8e^{2x}\right)}{2}\,dx = -\int 4x\sin(2x)\,dx - 4\int e^{2x}\sin(2x)\,dx =$

$-\left[\sin(2x) - 2x\cos(2x)\right] - 4\left[\dfrac{e^{2x}}{2^2 + 2^2}\left(2\sin(2x) - 2\cos(2x)\right)\right] = -\sin(2x) + 2x\cos(2x) - e^{2x}\sin(2x) + e^{2x}\cos(2x)$

using formulas 38 (with $u = 2x$) and 49 from the table of integrals. $\boxed{4 \text{ pts.}}$

$u_2 = \displaystyle\int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{\cos(2x)\left(8x + 8e^{2x}\right)}{2}\,dx = \int 4x\cos(2x)\,dx + 4\int e^{2x}\cos(2x)\,dx =$

$$\cos(2x) + 2x\sin(2x) + 4\left[\frac{e^{2x}}{2^2+2^2}\left(2\cos(2x)+2\sin(2x)\right)\right] = \cos(2x) + 2x\sin(2x) + e^{2x}\cos(2x) + e^{2x}\sin(2x)$$

using formulas 39 (with $u = 2x$) and 50 from the table of integrals. $\boxed{\text{4 pts.}}$

Therefore, $y_p = u_1y_1 + u_2y_2 = \left[-\sin(2x) + 2x\cos(2x) - e^{2x}\sin(2x) + e^{2x}\cos(2x)\right]\cos(2x)+$

$\left[\cos(2x) + 2x\sin(2x) + e^{2x}\cos(2x) + e^{2x}\sin(2x)\right]\sin(2x) =$
$-\sin(2x)\cos(2x) + 2x\cos^2(2x) - e^{2x}\sin(2x)\cos(2x) + e^{2x}\cos^2(2x)+$
$\cos(2x)\sin(2x) + 2x\sin^2(2x) + e^{2x}\cos(2x)\sin(2x) + e^{2x}\sin^2(2x) =$
$\left(2x + e^{2x}\right)\left(\cos^2(2x) + \sin^2(2x)\right) = 2x + e^{2x}$ $\boxed{\text{5 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1\cos(2x) + c_2\sin(2x) + 2x + e^{2x}$. $\boxed{\text{3 pts.}}$

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution. $y = c_1\cos(2x) + c_2\sin(2x) + 2x + e^{2x} \Rightarrow y' = -2c_1\sin(2x) + 2c_2\cos(2x) + 2 + 2e^{2x}$.
$y(0) = 1 \Rightarrow 1 = c_1\cos(0) + c_2\sin(0) + 2(0) + e^0 = c_1 + 1 \Rightarrow c_1 = 0$
$y'(0) = 2 \Rightarrow 2 = -2c_1\sin(0) + 2c_2\cos(0) + 2 + 2e^0 = 2c_2 + 4 \Rightarrow 2c_2 = -2 \Rightarrow c_2 = -1$ $\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = -\sin(2x) + 2x + e^{2x}}}$

Problem 3. (20 points) Consider a forced, damped mass-spring system with mass $m = 1$ kg, damping constant $c = 10$ N·s/m, spring constant $k = 16$ N/m, and external force $F_{\text{ext}} = 80\cos(4t)$ N. Find the steady-state (steady periodic) solution x_{sp}.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_{\text{e}}(t)$. $\boxed{\text{2 pts.}}$
In this problem, the d.e. becomes $x'' + 10x' + 16x = 80\cos(4t)$. $\boxed{\text{2 pts.}}$

The steady periodic solution is the particular solution x_p. $\boxed{\text{4 pts.}}$ Since the nonhomogeneous term $80\cos(4t)$ is a cosine, we should guess that x_p is a combination of a cosine and a sine with the same frequency: $x_p = A\cos(4t) + B\sin(4t)$. $\boxed{\text{5 pts.}}$ (No part of this guess will duplicate part of x_c because x_c is a transient term containing decaying exponential functions.)
$x = A\cos(4t) + B\sin(4t) \Rightarrow x' = -4A\sin(4t) + 4B\cos(4t) \Rightarrow x'' = -16A\cos(4t) - 16B\sin(4t)$.
Therefore, the left side of the d.e. is
$x'' + 10x' + 16x = -16A\cos(4t) - 16B\sin(4t) + 10\left[-4A\sin(4t) + 4B\cos(4t))\right] + 16\left[A\cos(4t) + B\sin(4t)\right]$
$= 40B\cos(3t) - 40A\sin(3t)$.
We want this to equal the nonhomogeneous term $80\cos(4t)$:
$40B\cos(4t) - 40A\sin(4t) = 80\cos(4t) \Rightarrow 40B = 80,\ -40A = 0 \Rightarrow A = 0$ and $B = 2$. Therefore,
$\boxed{\boxed{x_{\text{sp}} = 2\sin(4t)}}$. $\boxed{\text{7 pts.}}$

Problem 4. (20 points) Solve the system $\begin{cases} x' &= 3y \\ y' &= 4x + y \end{cases}$

Take the derivative of both sides of the first d.e. in the system: $x' = 3y \Rightarrow x'' = 3y'$. The second d.e. in the system is $y' = 4x + y$. Therefore, $x'' = 3(4x + y) = 12x + 3y$. From the first d.e. in the system, $3y = x'$, so we have $x'' = 12x + x'$ $\boxed{\text{8 pts.}}$ $x'' = 12x + x' \Rightarrow x'' - x' - 12x = 0$.
Char. eqn.: $r^2 - r - 12 = 0 \Rightarrow (r+3)(r-4) = 0 \Rightarrow r = -3$ or $r = 4 \Rightarrow x = c_1e^{-3t} + c_2e^{4t}$. $\boxed{\text{8 pts.}}$

The first d.e. in the given system says $y = x'/3$, so $y = \left(-3c_1e^{-3t} + 4c_2e^{4t}\right)/3$. Therefore, the solution of the given system is $\boxed{\boxed{x = c_1e^{-3t} + c_2e^{4t},\ y = -c_1e^{-3t} + \frac{4}{3}c_2e^{3t}}}$ $\boxed{\text{4 pts.}}$