MATH.2360 Engineering Differential Equations
Solving First Order Equations

1. Write the d.e. in the form $\dfrac{dy}{dx} = f(x, y)$.

2. Can the right-hand side $f(x, y)$ be factored into a term involving only x times a term involving only y: $\dfrac{dy}{dx} = g(x)h(y)$? If so, the d.e. is **separable**. Separate variables and integrate:

$$\int \frac{dy}{h(y)} = \int g(x)\ dx$$

3. Can the right-hand side $f(x, y)$ be written as $(\text{blob}_1)\, y + \text{blob}_2$, where the blobs involve only x and constants? If so, the d.e. is **linear**. Rewrite the d.e. in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

multiply both sides of the standard form of the d.e. by the integrating factor $\rho(x) = e^{\int P(x)\ dx}$, rewrite the d.e. as $\dfrac{d}{dx}\left[\rho(x)y\right] = \rho(x)Q(x)$, then integrate to get

$$\rho(x)y = \int \rho(x)Q(x)\ dx + c.$$

4. Can the right-hand side $f(x, y)$ be written as a function of y/x? For example, can $f(x, y)$ be written as the ratio of two polynomials, each of whose terms all have the same degree? If so, the d.e. is **homogeneous**. Let $v = y/x$. Replace y by xv and replace dy/dx by $v + x\dfrac{dv}{dx}$. The new d.e. is separable. Solve for v, then replace v by y/x.

5. Rewrite the d.e. in the form $M(x, y) + N(x, y)\dfrac{dy}{dx} = 0$. If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is **exact**. Let $f(x, y) = \int M(x, y)\ dx$, remembering that the "constant" of integration is $g(y)$. Compute $\dfrac{\partial f}{\partial y}$, set it equal to $N(x, y)$, solve for $g'(y)$, then integrate to find $g(y)$. Plug this expression for $g(y)$ into your formula for $f(x, y)$. The solution of the d.e. is

$$f(x, y) = c.$$