Engineering Differential Equations
Example using Laplace Transform to Solve a DE with Impulse Forcing Term

Problem: Solve the IVP $x'' + x = \delta(t-5),\ x(0) = 0,\ x'(0) = 0.$

Step 1. Take the transform of both sides of the d.e:

$$\mathcal{L}\{x'' + x\} = \mathcal{L}\{\delta(t-5)\}$$

Step 2. Solve for $\mathcal{L}\{x\}$:

$$\mathcal{L}\{x''\} + \mathcal{L}\{x\} = e^{-5s} \Rightarrow$$
$$s^2\mathcal{L}\{x\} - sx(0) - x'(0) + \mathcal{L}\{x\} = e^{-5s} \Rightarrow$$
$$s^2\mathcal{L}\{x\} - s \cdot 0 - 0 + \mathcal{L}\{x\} = e^{-5s} \Rightarrow$$
$$(s^2+1)\mathcal{L}\{x\} = e^{-5s} \Rightarrow$$
$$\mathcal{L}\{x\} = e^{-5s}\frac{1}{(s^2+1)}$$

Step 3. Solve for x:

$$x = \mathcal{L}^{-1}\left\{e^{-5s}\frac{1}{(s^2+1)}\right\} \Rightarrow$$
$$x = u(t-5)f(t-5),$$
where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin(t) \Rightarrow$
$$x = u(t-5)\sin(t-5).$$