Problem 1. Solve the following initial value problem: $xy' - \dfrac{y^2}{x^2} = 0,\ y(1) = 1.$

This is a separable d.e.

$$xy' - \frac{y^2}{x^2} = 0 \Rightarrow x\frac{dy}{dx} = \frac{y^2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{y^2}{x^3} \Rightarrow dy = \frac{y^2}{x^3}\,dx \Rightarrow \frac{dy}{y^2} = \frac{dx}{x^3}$$

$$\Rightarrow \int y^{-2}\,dy = \int x^{-3}\,dx \Rightarrow \frac{y^{-1}}{-1} = \frac{x^{-2}}{-2} + c \Rightarrow \frac{1}{y} = \frac{1}{2x^2} - c$$

$$y(1) = 1 \Rightarrow \frac{1}{1} = \frac{1}{2(1)^2} - c \Rightarrow c = -\frac{1}{2}\ \boxed{\text{1 pt.}} \Rightarrow \frac{1}{y} = \frac{1}{2x^2} + \frac{1}{2} = \frac{1+x^2}{2x^2} \Rightarrow \boxed{\boxed{y = \frac{2x^2}{x^2+1}}}$$

Problem 2. Solve the following initial value problem: $xyy' + y^2 - x^2 = 0,\ y(2) = 1.$

$xyy' + y^2 - x^2 = 0 \Rightarrow y' = \dfrac{x^2 - y^2}{xy}$. Since y' equals a rational function in which each term has the same degree (2), the d.e. is homogeneous.

We introduce the new variable $v = y/x$. In the d.e. we replace y' by $v + x\dfrac{dv}{dx}$ and we replace y by xv:

$$y' = \frac{x^2 - y^2}{xy} \Rightarrow v + x\frac{dv}{dx} = \frac{x^2 - (xv)^2}{x(xv)} = \frac{x^2\left(1 - v^2\right)}{x^2 v} = \frac{1 - v^2}{v} \Rightarrow x\frac{dv}{dx} = \frac{1 - v^2}{v} - v = \frac{1 - 2v^2}{v}$$

$$\Rightarrow \underbrace{\int \frac{v}{1 - 2v^2}\,dv}_{\text{Let } u = 1 - 2v^2} = \int \frac{dx}{x} \Rightarrow -\frac{1}{4}\ln\left(1 - 2v^2\right) = \ln(x) + c \Rightarrow e^{-\frac{1}{4}\ln\left(1 - 2v^2\right)} = e^{\ln(x) + c} \Rightarrow$$

$$\left(1 - 2v^2\right)^{-1/4} = e^{\ln(x)}\underbrace{e^c}_{c_1} = c_1 x \Rightarrow 1 - 2v^2 = \underbrace{c_1^{-4}}_{c_2} x^{-4} = \frac{c_2}{x^4} \Rightarrow 1 - 2(y/x)^2 = \frac{c_2}{x^4}$$

$$y(2) = 1 \Rightarrow 1 - 2(1/2)^2 = \frac{c_2}{2^4} \Rightarrow \frac{1}{2} = \frac{c_2}{16} \Rightarrow c_2 = 8$$

Therefore, $1 - 2(y/x)^2 = \dfrac{8}{x^4} \Rightarrow x^4 - 2x^2y^2 = 8 \Rightarrow \boxed{\boxed{y = \dfrac{\sqrt{x^4 - 8}}{\sqrt{2}x}}}$.

Problem 3. Solve the following initial value problem: $y' - \dfrac{4y}{x} = x^4\cos(x),\ y(\pi) = 0.$

This is a linear d.e. because y and y' appear to the first power and are multiplied by constants or functions of x.

The d.e. is already in standard form.

Find the integrating factor: $\rho(x) = e^{\int P(x)\ dx} = e^{\int -4/x\ dx} = e^{-4\ln(x)} = x^{-4}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$$x^{-4}\left[\frac{dy}{dx} - \left(\frac{4}{x}\right)y\right] = x^{-4}\left(x^4\cos(x)\right) = \cos(x) \Rightarrow x^{-4}\frac{dy}{dx} - 4x^{-5}y = \cos(x).$$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\left[x^{-4}y\right] = \cos(x)$.

Integrating both sides, we obtain $x^{-4}y = \int \cos(x)\ dx = \sin(x) + c$.

$y(\pi) = 0 \Rightarrow \pi^{-4}(0) = \sin(\pi) + c = c \Rightarrow c = 0$

Therefore, $x^{-4}y = \sin(x)$, so $\boxed{\boxed{y = x^4 \sin(x).}}$

Problem 4. Solve the following initial value problem: $2xyy' + y^2 - 4x^3 = 0,\ y(1) = 2$.

The given d.e. is exact: $\underbrace{2xy}_{N} y' + \underbrace{y^2 - 4x^3}_{M} = 0$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[y^2 - 4x^3\right] = 2y$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}[2xy] = 2y$. The d.e. is exact because $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Therefore, the solution of the d.e. is $f(x, y) = c$ where f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = y^2 - 4x^3$ and $\dfrac{\partial f}{\partial y} = N = 2xy$.

$\dfrac{\partial f}{\partial x} = y^2 - 4x^3 \Rightarrow f = \int \left(y^2 - 4x^3\right)\ \partial x = xy^2 - x^4 + g(y) \Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[xy^2 - x^4 + g(y)\right] = 2xy + g'(y)$.

But $\dfrac{\partial f}{\partial y} = N = 2xy$. Therefore, $2xy + g'(y) = 2xy \Rightarrow g'(y) = 0 \Rightarrow g(y) = \int 0\ dy = 0$ so $f = xy^2 - x^4$.The solution of the d.e. is $xy^2 - x^4 = c$

$y(1) = 2 \Rightarrow 1(2)^2 - 1^4 = c \Rightarrow c = 3 \Rightarrow xy^2 - x^4 = 3 \Rightarrow y^2 = \dfrac{x^4 + 3}{x} \Rightarrow \boxed{\boxed{y = \sqrt{\dfrac{x^4 + 3}{x}}}}$

Problem 5. Let P denote the population of a colony of tribbles. Suppose that β (the number of births per week per tribble) is proportional to $\sqrt{P}$ and that δ (the number of deaths per week per tribble) equals 0. Suppose the initial population is 4 and the population after 1 week is 9. What is the population after 2 weeks?

Recall that the de modeling population problems is $\dfrac{dP}{dt} = \beta P - \delta P$.

Let t denote time (in weeks). $\beta = k\sqrt{P}$ and $\delta = 0$ so $\dfrac{dP}{dt} = \left(k\sqrt{P}\right) P - 0 \cdot P = kP^{3/2}$

This d.e. is separable: $\dfrac{dP}{dt} = kP^{3/2} \Rightarrow \dfrac{dP}{P^{3/2}} = k\ dt \Rightarrow \int P^{-3/2}\ dP = \int k\ dt \Rightarrow \dfrac{P^{-1/2}}{-1/2} = kt + c$

$\Rightarrow -\dfrac{2}{\sqrt{P}} = kt + c$

$P(0) = 4 \Rightarrow -\dfrac{2}{\sqrt{4}} = k(0) + c \Rightarrow c = -1 \Rightarrow -\dfrac{2}{\sqrt{P}} = kt - 1 \Rightarrow \dfrac{2}{\sqrt{P}} = 1 - kt$

$P(1) = 9 \Rightarrow \dfrac{2}{\sqrt{9}} = 1 - k(1) \Rightarrow \dfrac{2}{3} = 1 - k \Rightarrow k = \dfrac{1}{3} \Rightarrow \dfrac{2}{\sqrt{P}} = 1 - \dfrac{t}{3}$

Therefore, $\dfrac{2}{\sqrt{P(2)}} = 1 - \dfrac{2}{3} = \dfrac{1}{3} \Rightarrow \dfrac{4}{P(2)} = \dfrac{1}{9} \Rightarrow \boxed{\boxed{P(2) = 36 \text{ tribbles}}}$

Problem 6. Find the general solution to each of the following linear homogeneous differential equations:

a. $y''' + 2y'' + 2y' = 0$

The characteristic equation is $r^3 + 2r^2 + 2r = 0 \Rightarrow r\left(r^2 + 2r + 2\right) = 0$

$r^2 + 2r + 2 = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \dfrac{-2 \pm \sqrt{-4}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

Therefore, the roots of the characteristic equation are $r = 0$ and $r = -1 \pm i$ so the solution of the d.e. is

$y = c_1 e^{0x} + c_2 e^{-1x}\cos(1x) + c_3 e^{-1x}\sin(1x)$, or $\boxed{\boxed{y = c_1 + c_2 e^{-x}\cos(x) + c_3 e^{-x}\sin(x)}}$.

b. $y^{(4)} - 9y'' = 0$

The characteristic equation is $r^4 - 9r^2 = 0 \Rightarrow r^2\left(r^2 - 9\right) = 0 \Rightarrow r^2(r+3)(r-3) = 0 \Rightarrow$

$r = 0$ (double root), $r = -3$, or $r = 3$

Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{-3x} + c_4 e^{3x}$, or

$\boxed{\boxed{y = c_1 + c_2 x + c_3 e^{-3x} + c_4 e^{3x}}}$.

Problem 7. Consider a forced, damped mass-spring system with mass 1 kg, damping coefficient 2 Ns/m, spring constant 4 N/m, and an external force $F_{\text{ext}}(t) = 8\cos(2t)$ N. Find the steady-state periodic solution $x_{\text{sp}}(t)$.

The d.e. modeling this system is $mx'' + cx' + kx = F_e(t)$, or $x'' + 2x' + 4x = 8\cos(2t)$.

The steady-state periodic solution $x_{\text{sp}}(t)$ is the particular solution $x_p(t)$.

The nonhomogeneous term in the de is $8\cos(2t)$, a cosine function. We should therefore guess that x_p is a combination of a cosine function and a sine function with the same coefficient of t: $x_p = A\cos(2t) + B\sin(2t)$. This guess does not duplicate any term x_c because x_c is a transient term containing decaying exponential functions, so there is no need to modify this guess.

$x = A\cos(2t) + B\sin(2t) \Rightarrow x' = -2A\sin(2t) + 2B\cos(2t) \Rightarrow x'' = -4A\cos(2t) - 4B\sin(2t)$
Therefore, the left side of the d.e. is $x'' + 2x' + 4x = -4A\cos(2t) - 4B\sin(2t) + 2\left[-2A\sin(2t) + 2B\cos(2t)\right] + 4\left[A\cos(2t) + B\sin(2t)\right] = 4B\cos(2t) - 4A\sin(2t)$.
We want this to equal the nonhomogeneous term $8\cos(2t)$:

$4B\cos(2t) - 4A\sin(2t) = 8\cos(2t) \Rightarrow 4B = 8,\ -4A = 0 \Rightarrow A = 0,\ B = 2$. Thus, $\boxed{\boxed{x_{\text{sp}} = x_p = 2\sin(2t)}}$.

Problem 8. Consider an RLC circuit with inductance $L = 1$ henry, resistance $R = 5\Omega$, capacitance $C = 0.25$ farads, and applied voltage $E(t) = 20\cos(2t)$ volts. Suppose the initial charge on the capacitor $Q(0) = 1$ coul and the initial current in the circuit $Q'(0) = 0$ amps. Find the current in the circuit $I(t)$.

The d.e. modeling this system is $LQ'' + RQ' + \frac{Q}{C} = E(t)$, or $Q'' + 5Q' + 4Q = 20\cos(2t)$.

Step 1. Find Q_c by solving the d.e. $Q'' + 5Q' + 4Q = 0$.
Characteristic equation: $r^2 + 5r + 4 = 0 \Rightarrow (r+4)(r+1) = 0 \Rightarrow r = -4$ or $r = -1$
Therefore, $Q_c = c_1 e^{-4t} + c_2 e^{-t}$.

Step 2. Find Q_p. You can use either of the following methods.

Method 1: Undetermined Coefficients. The nonhomogeneous term in the de is $20\cos(2t)$, a cosine function. We should therefore guess that Q_p is a combination of a cosine function and a sine function with the same coefficient of t: $Q_p = A\cos(2t) + B\sin(2t)$. This guess does not duplicate any term Q_c, so there is no need to modify this guess.

$Q = A\cos(2t) + B\sin(2t) \Rightarrow Q' = -2A\sin(2t) + 2B\cos(2t) \Rightarrow Q'' = -4A\cos(2t) - 4B\sin(2t)$
Therefore, the left side of the d.e. is $Q''+5Q'+4Q = -4A\cos(2t)-4B\sin(2t)+5\left[-2A\sin(2t)+2B\cos(2t)\right]+4\left[A\cos(2t)+B\sin(2t)\right] = 10B\cos(2t) - 10A\sin(2t)$.
We want this to equal the nonhomogeneous term $20\cos(2t)$:
$10B\cos(2t) - 10A\sin(2t) = 20\cos(2t) \Rightarrow 10B = 20,\ -10A = 0 \Rightarrow A = 0,\ B = 2$. Thus, $Q_p = 2\sin(2t)$.

Method 2: Variation of Parameters. From Q_c we obtain two independent solutions of the homogeneous d.e: $Q_1 = e^{-4t}$ and $Q_2 = e^{-t}$. The Wronskian is given by

$$W(x) = \begin{vmatrix} Q_1 & Q_2 \\ Q_1' & Q_2' \end{vmatrix} = \begin{vmatrix} e^{-4t} & e^{-t} \\ -4e^{-4t} & -e^{-t} \end{vmatrix} = e^{-4t}\left(-e^{-t}\right) - \left(-4e^{-4t}\right)\left(e^{-t}\right) = 3e^{-5t}$$

$$u_1 = \int \frac{-Q_2 F_{\text{ext}}(t)}{W(t)}\,dt = -\int \frac{e^{-t}\left(20\cos(2t)\right)}{3e^{-5t}}\,dt = -\frac{20}{3}\int e^{5t}\cos(2t)\,dt =$$

$$-\frac{20}{3}\left[\frac{e^{4t}}{4^2+2^2}\left(4\cos(2t)+2\sin(2t)\right)\right] = e^{4t}\left[-\frac{4}{3}\cos(2t) - \frac{2}{3}\sin(2t)\right]$$ using formula 50 from the integral table.

$$u_2 = \int \frac{Q_1 F_{\text{ext}}(t)}{W(t)}\,dt = \int \frac{e^{-4t}\left(20\cos(2t)\right)}{3e^{-5t}}\,dt = \frac{20}{3}\int e^{t}\cos(2t)\,dt =$$

$$\frac{20}{3}\left[\frac{e^{t}}{1^2+2^2}\left(1\cos(2t)+2\sin(2t)\right)\right] = e^{t}\left[\frac{4}{3}\cos(2t) + \frac{8}{3}\sin(2t)\right]$$ using formula 50 from the integral table.

Therefore, $Q_p = u_1Q_1 + u_2Q_2 = e^{4t}\left[-\frac{4}{3}\cos(2t) - \frac{2}{3}\sin(2t)\right]\left(e^{-4t}\right) + e^{t}\left[\frac{4}{3}\cos(2t) + \frac{8}{3}\sin(2t)\right]\left(e^{-t}\right) = 2\sin(2t)$

Step 3. $Q = Q_c + Q_p$, so $Q = c_1e^{-4t} + c_2e^{-t} + 2\sin(2t)$

Step 4. Use the initial conditions to determine the values of c_1 and c_2.
$Q = c_1e^{-4t} + c_2e^{-t} + 2\sin(2t) \Rightarrow Q' = -4c_1e^{-4t} - c_2e^{-t} + 4\cos(2t)$
$Q(0) = 1 \Rightarrow c_1e^0 + c_2e^0 + 2\sin(0) = 1 \Rightarrow c_1 + c_2 = 1.$
$Q'(0) = 0 \Rightarrow -4c_1e^0 - c_2e^0 + 4\cos(0) = 0 \Rightarrow -4c_1 - c_2 = -4.$

$c_1 + c_2 = 1,\ -4c_1 - c_2 = -4 \Rightarrow c_1 = 1,\ c_2 = 0$

Therefore, $Q = e^{-4t} + 2\sin(2t)$ so $\boxed{I = Q' = -4e^{-4t} + 4\cos(2t)}$

Problem 9. Use the Laplace Transform to solve the following IVP: $x'' + 5x' + 6x = 4e^{-t},\ x(0) = 1,\ x'(0) = 0.$

Solutions not using the Laplace transform method will not receive any credit. x is a function of t. x'' means $\frac{d^2x}{dt^2}$.

$$x'' + 5x' + 6x = 4e^{-t} \Rightarrow \mathcal{L}\left\{x'' + 5x' + 6x\right\} = \mathcal{L}\left\{4e^{-t}\right\} \Rightarrow \mathcal{L}\left\{x''\right\} + 5\mathcal{L}\left\{x'\right\} + 6\mathcal{L}\{x\} = 4\mathcal{L}\left\{e^{-t}\right\} = \frac{4}{s+1}$$

$$\Rightarrow \left[s^2\mathcal{L}\{x\} - sx(0) - x'(0)\right] + 5\left[s\mathcal{L}\{x\} - x(0)\right] + 6\mathcal{L}\{x\} = \frac{4}{s+1}$$

$$\left[s^2\mathcal{L}\{x\} - s\cdot 1 - 0\right] + 5\left[s\mathcal{L}\{x\} - 1\right] + 6\mathcal{L}\{x\} = \frac{4}{s+1} \Rightarrow$$

$$s^2\mathcal{L}\{x\} - s + 5s\mathcal{L}\{x\} - 5 + 6\mathcal{L}\{x\} = \frac{4}{s+1} \Rightarrow \left(s^2 + 5s + 6\right)\mathcal{L}\{x\} = \frac{4}{s+1} + s + 5 = \frac{s^2+6s+9}{s+1} \Rightarrow$$

$$\mathcal{L}\{x\} = \frac{s^2+6s+9}{(s+1)\left(s^2+5s+6\right)} \Rightarrow x = \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s+1)\left(s^2+5s+6\right)}\right\}.$$

Use a partial fraction decomposition:

$$\frac{s^2+6s+9}{(s+1)\left(s^2+5s+6\right)} = \frac{(s+3)^2}{(s+1)(s+2)(s+3)} = \frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$(s+1)(s+2)\left[\frac{s+3}{(s+1)(s+2)}\right] = (s+1)(s+2)\left[\frac{A}{s+1} + \frac{B}{s+2}\right] \Rightarrow$$

$$s+3 = A(s+2) + B(s+1) = (A+B)s + (2A+B)$$

$$\Rightarrow A+B=1,\ 2A+B=3 \Rightarrow A=2,\ B=-1.$$

Therefore, $x = \mathcal{L}^{-1}\left\{\frac{2}{s+1} + \frac{-1}{s+2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \Rightarrow \boxed{\boxed{x = 2e^{-t} - e^{-2t}}}$