MATH.2360 Engineering Differential Equations
Solutions to Sample Problems for Exam 1

Problem 1. Is $y = x^2$ a solution of the d.e. $x^2y' = y^2 + x^3$? Why or why not?

Left side of d.e: $y = x^2 \Rightarrow y' = 2x \Rightarrow x^2y' = x^2(2x) = 2x^3$.

Right side of d.e: $y = x^2 \Rightarrow y^2 + x^3 = \left(x^2\right)^2 + x^3 = x^4 + x^3$.

Left side $\neq$ right side, so $\boxed{\boxed{y(x) = x^2 \textbf{ is not} \text{ a solution of the d.e. } x^2y' = y^2 + x^3}}$

Problem 2. Is $y(x) = x^3$ is a solution of the d.e. $x^4y' = x^6 + 2y^2$? Why or why not?

Left side of d.e: $y(x) = x^3 \Rightarrow y' = 3x^2 \Rightarrow x^4y' = x^4\left(3x^2\right) = 3x^6$.

Right side of d.e: $y = x^3 \Rightarrow x^6 + 2y^2 = x^6 + 2\left(x^3\right)^2 = x^6 + 2x^6 = 3x^6$.

Left side = right side, so $\boxed{\boxed{y(x) = x^2 \textbf{ is} \text{ a solution of the d.e. } x^4y' = x^6 + 2y^2}}$

Problem 3. A car is traveling at a speed of 20 m/s when the driver applies the brakes. The car stops in 5 seconds. How far (in meters) does the car travel in that time? Assume the car's deceleration is constant.

For objects moving in one dimension under constant acceleration, the velocity and position of the object are given by the formulas $v = at + v_0$ and $x = \frac{1}{2}at^2 + v_0t + x_0$. Take $t = 0$ to be the time the driver applies the brakes, and take $x = x_0 = 0$ to be the position at which the driver applies the brakes. We are told that $v_0 = 20$, so $v = at + 20$ and $x = \frac{1}{2}at^2 + 20t$. At $t = 5$ we have $v = 0$, so $0 = a(5) + 20 \Rightarrow a = -4$. Therefore, $x = \frac{1}{2}(-4)t^2 + 20t = -2t^2 + 20t$, so at $t = 5$ we have $\boxed{\boxed{x = -2(5)^2 + 20(5) = 50 \text{ m}}}$.

Problem 4. A container initially holds 50 grams of a radioactive substance. There are 45 grams of the substance left after 2 hours. When will there be 20 grams of the substance left in the container?

Let t denote time in hours, and let x denote the amount (in grams) of radioactive substance in the container at time t. Then $x = x_0e^{-kt}$, where $x_0 = x(0)$.

We are told that $x_0 = 50$ so $x = 50e^{-kt}$

$x(2) = 45 \Rightarrow 45 = 50e^{-k(2)} = e^{-2k} \Rightarrow 45/50 = e^{-2k} \Rightarrow \ln(0.9) = \ln\left(e^{-2k}\right) = -2k \Rightarrow k = -\ln(0.9)/2$

To find the time when $x = 20$, set $x = 20$ in the equation $x = 50e^{-kt}$ and solve for t:

$$20 = 50e^{-kt} \Rightarrow 20/50 = e^{-kt} \Rightarrow \ln(0.4) = \ln\left(e^{-kt}\right) = -kt \Rightarrow t = \frac{\ln(0.4)}{-k} \Rightarrow \boxed{\boxed{t = 2\frac{\ln(0.4)}{\ln(0.9)} \approx 17.4 \text{ hours}}}.$$

Problem 5. Solve the following initial value problem.

$$2y\frac{dy}{dx} = x\left(y^2+1\right), \quad y(2) = 0.$$

This is a separable d.e.

$2y\dfrac{dy}{dx} = x\left(y^2+1\right) \Rightarrow 2y\ dy = x\left(y^2+1\right)\ dx \Rightarrow \dfrac{2y}{y^2+1}\ dy = x\ dx.$

$\Rightarrow \underbrace{\displaystyle\int \frac{2y}{y^2+1}\ dy}_{u=y^2+1} = \displaystyle\int x\ dx \Rightarrow \ln\left(y^2+1\right) = \dfrac{x^2}{2} + c.$

$y(2) = 0 \Rightarrow \ln\left(0^2+1\right) = \dfrac{2^2}{2} + c \Rightarrow 0 = 2 + c \Rightarrow c = -2 \Rightarrow \boxed{\boxed{\ln\left(y^2+1\right) = \dfrac{x^2}{2} - 2}}.$

Problem 6. Solve the following initial value problem.

$$x\frac{dy}{dx} = 2x^2 + y, \quad y(1) = 3.$$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone.

First write the equation in standard form:

$x\dfrac{dy}{dx} = 2x^2 + y \Rightarrow \dfrac{dy}{dx} = 2x + \dfrac{y}{x} \Rightarrow \dfrac{dy}{dx} - \left(\dfrac{1}{x}\right)y = 2x$

Next, find the integrating factor: $\rho(x) = e^{\int -1/x\ dx} = e^{-\ln(x)} = x^{-1}.$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^{-1}\left[\dfrac{dy}{dx} - \left(\dfrac{1}{x}\right)y\right] = x^{-1}(2x) \Rightarrow x^{-1}\dfrac{dy}{dx} - x^{-2}y = 2.$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\left[x^{-1}y\right] = 2.$

Integrating both sides, we obtain $x^{-1}y = \displaystyle\int 2\ dx = 2x + c.$

$y(1) = 3 \Rightarrow 1^{-1}(3) = 2(1) + c \Rightarrow c = 1$

Therefore, $x^{-1}y = 2x + 1$, so $\boxed{\boxed{y = 2x^2 + x.}}$

Problem 7. Solve the following initial value problem.

$$\frac{dy}{dx} = \frac{x^2+2y}{x}, \quad y(1) = 2.$$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone.

First write the equation in standard form:

$\dfrac{dy}{dx} = \dfrac{x^2+2y}{x} = x + \dfrac{2y}{x} \Rightarrow \dfrac{dy}{dx} - \left(\dfrac{2}{x}\right)y = x$

Next, find the integrating factor: $\rho(x) = e^{\int -2/x \, dx} = e^{-2\ln(x)} = x^{-2}$.

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^{-2}\left[\frac{dy}{dx} - \left(\frac{2}{x}\right) y\right] = x^{-2}(x) \Rightarrow x^{-2}\frac{dy}{dx} - 2x^{-3}y = x^{-1}$.

Use the Product Rule backwards to rewrite the d.e. as $\frac{d}{dx}\left[x^{-2}y\right] = x^{-1}$.

Integrating both sides, we obtain $x^{-2}y = \int x^{-1}\, dx = \ln(x) + c$.

$y(1) = 2 \Rightarrow 1^{-2}(2) = \ln(1) + c \Rightarrow c = 2$

Therefore, $x^{-2}y = \ln(x) + 2$, so $\boxed{\boxed{y = x^2\ln(x) + 2x^2.}}$

Problem 8. Solve the following initial value problem.

$$\frac{dy}{dx} = \frac{2x}{1+2y}, \quad y(2) = 1.$$

This is a separable d.e.

$\frac{dy}{dx} = \frac{2x}{1+2y} \Rightarrow (1+2y)\ dy = 2x\ dx.$

$\Rightarrow \int (1+2y)\ dy = \int 2x\ dx \Rightarrow y + y^2 = x^2 + c.$

$y(2) = 1 \Rightarrow 1 + 1^2 = 2^2 + c \Rightarrow c = -2 \Rightarrow y + y^2 = x^2 - 2 \Rightarrow \boxed{\boxed{y = \frac{-1+\sqrt{4x^2-7}}{2}}}.$

Problem 9. A tank initially contains 100 liters of water in which 50 grams of salt are dissolved. A salt solution containing 10 grams of salt per liter is pumped into the tank at the rate of 4 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 6 liters per minute.

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in grams). Write down the differential equation $\left(\frac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.

$\frac{dx}{dt} = \text{rate in - rate out}$

= (flow rate in)(concentration in) - (flow rate out)(concentration out), so

$\frac{dx}{dt} = \left(4\frac{\text{L}}{\text{min}}\right)\left(10\frac{\text{gm}}{\text{L}}\right) - \left(6\frac{\text{L}}{\text{min}}\right)\left(\frac{x\ \text{gm}}{(100-2t)\ \text{L}}\right).$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 100 + (4-6)t$ liters.)

Initially there are 50 gm. of salt in the tank, so $x(0) = 50$.

Therefore, the initial value problem describing this mixing problem is $\boxed{\boxed{\frac{dx}{dt} = 40 - \frac{6x}{100-2t} \quad \text{with } x(0) = 50.}}$