

MATH.2360 Engineering Differential Equations
Solutions to Sample Problems for Exam # 2

Problem 1. Consider the autonomous differential equation $\frac{dx}{dt} = 4x^2 - x^4$.

- a. Find all critical points (equilibrium solutions) of this d.e.

$$4x^2 - x^4 = 0 \Rightarrow x^2(4 - x^2) = 0 \Rightarrow x^2(2 + x)(2 - x) = 0 \Rightarrow$$

the critical points are $-2, 0$ and 2

- b. Draw the phase line (phase diagram) for this d.e.

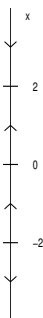
The three critical points divide the phase line into 4 intervals: $x > 2$, $0 < x < 2$, $-2 < x < 0$, and $x < -2$.

$$\left. \frac{dx}{dt} \right|_{x=3} = 3^2(2+3)(2-3) < 0, \text{ so the direction arrow points down for } x > 2.$$

$$\left. \frac{dx}{dt} \right|_{x=1} = 1^2(2+1)(2-1) > 0, \text{ so the direction arrow points up for } 0 < x < 2.$$

$$\left. \frac{dx}{dt} \right|_{x=-1} = (-1)^2(2-1)(2-(-1)) > 0, \text{ so the arrow points up for } -2 < x < 0.$$

$$\left. \frac{dx}{dt} \right|_{x=-3} = (-3)^2(2-3)(2-(-3)) < 0, \text{ so the arrow points down for } x < -2.$$



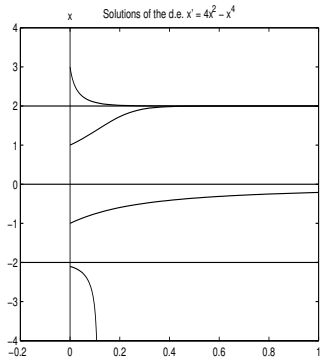
- c. Determine whether each critical point is stable or unstable.

From the phase line we can see that 2 is stable but -2 and 0 are unstable.

- d. If $x(0) = -1$, what value will $x(t)$ approach as t increases?

Since -1 lies in the interval $-2 < x < 0$, we can see from the phase line that $x(t) \rightarrow 0$ as t increases.

- e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes.



Problem 2. Solve the following differential equations:

a. $y'' - 10y' + 25y = 0$

Characteristic equation: $r^2 - 10r + 25 = 0 \Rightarrow (r - 5)^2 = 0 \Rightarrow$

$r = 5$ (repeated root). Therefore, $y = c_1 e^{5x} + c_2 x e^{5x}$

b. $y'' + y' - 2y = 0$

Characteristic equation: $r^2 + r - 2 = 0 \Rightarrow (r + 2)(r - 1) = 0 \Rightarrow$

$r = -2$ or $r = 1$. Therefore, $y = c_1 e^{-2x} + c_2 e^x$

c. $y'' + 6y' + 8y = 0$

Characteristic equation: $r^2 + 6r + 8 = 0 \Rightarrow (r + 4)(r + 2) = 0 \Rightarrow$

$r = -4$ or $r = -2$. Therefore, $y = c_1 e^{-4x} + c_2 e^{-2x}$

d. $y'' - 6y' + 9y = 0$

Characteristic equation: $r^2 - 6r + 9 = 0 \Rightarrow (r - 3)^2 = 0 \Rightarrow$

$r = 3$ (repeated root). Therefore, $y = c_1 e^{3x} + c_2 x e^{3x}$

Problem 3. Solve the following initial value problem.

$$3x^2 + 6xy + [3x^2 + 2y] \frac{dy}{dx} = 0, \quad y(1) = 0.$$

$$\underbrace{3x^2 + 6xy}_M + \underbrace{[3x^2 + 2y]}_N \frac{dy}{dx} = 0.$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [3x^2 + 6xy] = 6x. \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [3x^2 + 2y] = 6x.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the d.e. is exact. Therefore, the solution of the d.e. is $f(x, y) = c$, where the function f satisfies the conditions $\frac{\partial f}{\partial x} = M = 3x^2 + 6xy$ and $\frac{\partial f}{\partial y} = N = 3x^2 + 2y$. $\frac{\partial f}{\partial x} = 3x^2 + 6xy \Rightarrow$

$$f = \int (3x^2 + 6xy) dx = x^3 + 3x^2y + g(y) \Rightarrow \frac{\partial f}{\partial y} = 3x^2 + g'(y).$$

But $\frac{\partial f}{\partial y} = N = 3x^2 + 2y$, so $3x^2 + g'(y) = 3x^2 + 2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2$.

Therefore, $f = x^3 + 3x^2y + y^2$, so the solution of the d.e. is $x^3 + 3x^2y + y^2 = c$.

The initial condition $y(1) = 0 \Rightarrow 1^3 + 3(1)^2(0) + 0^2 = c \Rightarrow c = 1$. Therefore, the solution of the given IVP is $\boxed{x^3 + 3x^2y + y^2 = 1}$

Problem 4. Solve the following initial value problem.

$$x^3 + y^3 - xy^2 \frac{dy}{dx} = 0, \quad y(1) = 3.$$

$x^3 + y^3 - xy^2 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$. Since dy/dx equals a rational function in which each term has the same degree (3), the d.e. is homogeneous. We introduce the new variable $v = y/x$. In the d.e. we replace $\frac{dy}{dx}$ by $v + x \frac{dv}{dx}$ and we replace y by xv :

$$\begin{aligned} \frac{dy}{dx} = \frac{x^3 + y^3}{xy^2} &\Rightarrow v + x \frac{dv}{dx} = \frac{x^3 + (xv)^3}{x(xv)^2} = \frac{x^3(1 + v^3)}{x^3v} = \frac{1 + v^3}{v^2} = \frac{1}{v^2} + v \Rightarrow x \frac{dv}{dx} = \frac{1}{v^2} \\ \Rightarrow v^2 dv = \frac{1}{x} dx &\Rightarrow \int v^2 dv = \int \frac{1}{x} dx \Rightarrow \frac{v^3}{3} = \ln(x) + c \Rightarrow \frac{(y/x)^3}{3} = \ln(x) + c \Rightarrow \end{aligned}$$

$(y/x)^3 = 3 \ln(x) + \underbrace{3c}_{c_1}$. The initial condition $y(1) = 3 \Rightarrow (3/1)^3 = 3 \ln(1) + c_1 \Rightarrow c_1 = 27$ 1 pt.

Therefore, $(y/x)^3 = 3 \ln(x) + 27 \Rightarrow (y/x) = [3 \ln(x) + 27]^{1/3} \Rightarrow \boxed{y = x [3 \ln(x) + 27]^{1/3}}$.

Problem 5. Solve the following initial value problem.

$$x^4 + 4y^4 - 4xy^3 \frac{dy}{dx} = 0, \quad y(1) = 2.$$

$x^4 + 4y^4 - 4xy^3 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x^4 + 4y^4}{4xy^3}$. dy/dx equals a rational function, and every term has the same degree (4). Therefore, this d.e. is homogeneous.

We introduce the new variable $v = y/x$. In the d.e. we replace $\frac{dy}{dx}$ by $v + x \frac{dv}{dx}$ and we replace y by xv :

$$\begin{aligned} \frac{dy}{dx} = \frac{x^4 + 4y^4}{4xy^3} &\Rightarrow v + x \frac{dv}{dx} = \frac{x^4 + 4(xv)^4}{4x(xv)^3} = \frac{x^4(1 + 4v^4)}{4x^4v^3} = \frac{1 + 4v^4}{4v^3} = \frac{1}{4v^3} + v \Rightarrow x \frac{dv}{dx} = \frac{1}{4v^3} \\ \Rightarrow 4v^3 dv = \frac{1}{x} dx &\Rightarrow \int 4v^3 dv = \int \frac{1}{x} dx \Rightarrow v^4 = \ln(x) + c \Rightarrow \left(\frac{y}{x}\right)^4 = \ln(x) + c \end{aligned}$$

The initial condition $y(1) = 2 \Rightarrow (2/1)^4 = \ln(1) + c \Rightarrow c = 16$.

Therefore, $\left(\frac{y}{x}\right)^4 = \ln(x) + 16 \Rightarrow \frac{y}{x} = [\ln(x) + 16]^{1/4} \Rightarrow \boxed{y = x [\ln(x) + 16]^{1/4}}$.

Problem 6. (20 points) Solve the following initial value problem.

$$2xy^2 + 3x^2 + (2x^2y + 4y^3) \frac{dy}{dx} = 0, \quad y(1) = 2.$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{2xy^2 + 3x^2}_M + \underbrace{(2x^2y + 4y^3)}_N \frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [2xy^2 + 3x^2] = 4xy. \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [2x^2y + 4y^3] = 4xy.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the d.e. is exact. Therefore, the solution of the d.e. is $f(x, y) = c$, where the

function f satisfies the conditions $\frac{\partial f}{\partial x} = M = 2xy^2 + 3x^2$ and $\frac{\partial f}{\partial y} = N = 2x^2y + 4y^3$.

$$\frac{\partial f}{\partial x} = 2xy^2 + 3x^2 \Rightarrow f = \int (2xy^2 + 3x^2) \partial x = x^2y^2 + x^3 + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [x^2y^2 + x^3 + g(y)] = 2x^2y + g'(y)$$

$$\text{But } \frac{\partial f}{\partial y} = N = 2x^2y + 4y^3 \Rightarrow 2x^2y + g'(y) = 2x^2y + 4y^3 \Rightarrow g'(y) = 4y^3 \Rightarrow g(y) = y^4 \Rightarrow f = x^2y^2 + x^3 + y^4$$

Therefore, the solution of the d.e. is $x^2y^2 + x^3 + y^4 = c$
 $y(1) = 2 \Rightarrow 1^2 \cdot 2^2 + 1^3 + 2^4 = c \Rightarrow c = 21$.

Therefore, the solution of the initial value problem is $\boxed{x^2y^2 + x^3 + y^4 = 21}$

Problem 7. (10 points) Let P denote the population of a colony of dodos. Suppose that the birth rate β (number of births per week per dodo) equals 0 and that the death rate δ (number of deaths per week per dodo) is constant. Suppose the initial population is 100, and after ten weeks the population is 50. What is the population after 43 weeks?

$$\frac{dP}{dt} = \beta P - \delta P = (0)P - (k)P = -kP.$$

This is a separable d.e: $\frac{dP}{dt} = -kP \Rightarrow \frac{dP}{P} = -k dt$

$$\Rightarrow \int P^{-1} dP = \int -k dt \Rightarrow \ln(P) = -kt + c \Rightarrow P = e^{-kt+c} = e^{-kt} \underbrace{e^c}_{c_1}$$

$$P(0) = 100 \Rightarrow 100 = e^0 c_1 \Rightarrow c_1 = 100$$

$$\Rightarrow P = 100e^{-kt}. P(10) = 50 \Rightarrow 50 = 100e^{-10k} \Rightarrow \frac{50}{100} = e^{-10k} \Rightarrow \ln(0.5) = \ln(e^{-10k}) = -10k \Rightarrow k = -\frac{\ln(0.5)}{10}.$$

Therefore, $P(43) = 100e^{-43k} \approx 5$ dodos.

Problem 8. (10 points) A ball of mass m falling vertically downward experiences two forces: its weight, and a drag force proportional to the *square* of its velocity. Let t denote time, and let v denote the ball's velocity at time t . (Assume $v < 0$ for the falling object. In other words, up is the positive direction.)

Write a differential equation ($dv/dt = \text{something}$) modeling the ball's motion. Make sure the drag force has the proper sign (+ or -). All parameters are assumed to be positive.

DO NOT SOLVE THE DIFFERENTIAL EQUATION, JUST WRITE IT DOWN.

Newton's Second Law says that $F = ma$, where F is the total force acting on an object and a is the object's acceleration.

The total force acting on the ball is $F = \underbrace{-mg}_{\text{weight}} + \underbrace{kv^2}_{\text{drag}}$

(Because the ball is falling, the drag is in the upward direction, so the drag force must be positive.)

Therefore, $F = ma \Rightarrow -mg + kv^2 = m \frac{dv}{dt} \Rightarrow \frac{dv}{dt} = -g + \underbrace{\frac{k}{m}}_{k_1} v^2 \Rightarrow \boxed{\frac{dv}{dt} = -g + k_1 v^2}$