92.236 Engineering Differential Equations Final Exam Solutions
Spring 2012

Problem 1. (10 pts.)

Solve the following initial value problem: $y' = \dfrac{xy^3}{\sqrt{1+x^2}}, \quad y(0) = -1.$

Note: y' means dy/dx.

This is a separable d.e. $\boxed{\text{2 pts.}}$ $\quad \dfrac{dy}{dx} = \dfrac{xy^3}{\sqrt{1+x^2}} \Rightarrow \dfrac{dy}{y^3} = \dfrac{xdx}{\sqrt{1+x^2}}$ $\boxed{\text{1 pt.}}$

$$\Rightarrow \int y^{-3}\, dy = \underbrace{\int x\left(1+x^2\right)^{-1/2}\, dx}_{u=1+x^2} \Rightarrow \frac{y^{-2}}{-2} = \left(1+x^2\right)^{1/2} + c \ \boxed{\text{6 pts.}}$$

$$\Rightarrow y^{-2} = -2\left(1+x^2\right)^{1/2} \underbrace{-2c}_{c_1} = c_1 - 2\left(1+x^2\right)^{1/2}$$

$y(0) = -1 \Rightarrow (-1)^{-2} = c_1 - 2\left(1+0^2\right)^{1/2} \Rightarrow c_1 = 3$ $\boxed{\text{1 pt.}}$

Therefore, $y^{-2} = 3 - 2\left(1+x^2\right)^{1/2} \Rightarrow \boxed{\boxed{y = -\left[3 - 2\left(1+x^2\right)^{1/2}\right]^{-1/2}.}}$

Problem 2. (10 pts.)

Solve the following initial value problem: $2xy^2 + 4 = \left(6 - 2x^2y\right)y'$ with $y(-1) = 8.$

Note: y' means dy/dx.

This d.e. is not separable, linear, or homogeneous. To see whether it is exact, move all terms to the same side of the equation: $2xy^2 + 4 - \left(6 - 2x^2y\right)y' = 0$, or $\underbrace{2xy^2+4}_{M} + \underbrace{\left(2x^2y - 6\right)}_{N}y' = 0$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[2xy^2 + 4\right] = 4xy$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[2x^2y - 6\right] = 4xy$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{2 pts.}}$

Therefore, the solution of the d.e. is $f(x,y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 2xy^2 + 4$ and $\dfrac{\partial f}{\partial y} = N = 2x^2y - 6.$

$\dfrac{\partial f}{\partial x} = 2xy^2 + 4 \Rightarrow f = \displaystyle\int \left(2xy^2 + 4\right)\, dx = x^2y^2 + 4x + g(y)$ $\boxed{\text{3 pts.}}$

$\Rightarrow \dfrac{\partial f}{\partial y} = 2x^2y + g'(y)$. But $\dfrac{\partial f}{\partial y} = N = 2x^2y - 6$, so $2x^2y + g'(y) = 2x^2y - 6$

$\Rightarrow g'(y) = -6 \Rightarrow g(y) = -6y$ $\boxed{\text{2 pts.}}$.

Therefore, $f = x^2y^2 + 4x - 6y$, so the solution of the d.e. is $x^2y^2 + 4x - 6y = c.$ $\boxed{\text{2 pts.}}$

The initial condition $y(-1) = 8 \Rightarrow (-1)^2 8^2 + 4(-1) - 6(8) = c \Rightarrow c = 12$ $\boxed{\text{1 pt.}}$.

Therefore, the solution of the given IVP is $x^2y^2 + 4x - 6y = 12$, or, using the quadratic formula to solve for y, $\boxed{\boxed{y = \dfrac{3 + \sqrt{9 + 12x^2 - 4x^3}}{x^2}}}$

Problem 3. (15 points)

A tank initially contains 12 liters of pure water (no salt). Water containing 2 grams of salt per liter is pumped into the tank at the rate of 3 liters per minute, and the well-mixed solution in the tank is pumped out of the tank at the rate of 3 liters per minute. How long will it take for the amount of salt in the tank to reach 12 grams?

let t denote time (in minutes) and let x denote the amount of salt in the tank at time t (in grams).

$\frac{dx}{dt}$ = rate in - rate out = (flow rate in)(concentration in) - (flow rate out)(concentration out),

$\boxed{\text{3 pts.}}$ so $\frac{dx}{dt} = \left(3\frac{\text{liters}}{\text{minute}}\right)\left(2\frac{\text{g}}{\text{liter}}\right) - \left(3\frac{\text{liters}}{\text{minute}}\right)\left(\frac{x \text{ g}}{12 \text{ liters}}\right)$ $\boxed{\text{3 pts.}}$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) = $12+(3-3)t =$ 12 liters.)

$$\frac{dx}{dt} = 6 - \frac{3x}{12} = 6 - \frac{x}{4} = -\frac{1}{4}(x-24) \Rightarrow \frac{dx}{x-24} = -\frac{1}{4}\,dt \Rightarrow \int \frac{dx}{x-24} = \int \left(-\frac{1}{4}\right)\,dt \Rightarrow$$

$$\ln|x-24| = -\frac{t}{4} + c \Rightarrow |x-24| = e^{-t/4+c} = e^{-t/4}e^{c} \Rightarrow x - 24 = \underbrace{\pm e^{c}}_{c_1} e^{-t/4} = c_1 e^{-t/4} \quad \boxed{\text{6 pts.}}$$

$x(0) = 0 \Rightarrow 0 - 24 = c_1 e^0 \Rightarrow c_1 = -24 \Rightarrow x = 24 - 24e^{-t/4}$ $\boxed{\text{2 pts.}}$

To find how long it takes for the amount of salt in the tank to reach 12 grams, set $x = 12$ and solve for t:

$$12 = 24 - 24e^{-t/4} \Rightarrow -12 = -24e^{-t/4} \Rightarrow 0.5 = e^{-t/4} \Rightarrow \ln(0.5) = \ln\left(e^{-t/4}\right) = -t/4 \Rightarrow$$

$\boxed{\boxed{t = -4\ln(0.5) \approx 2.8 \text{ minutes}}}$ $\boxed{\text{1 pt.}}$

Problem 4. (10 pts.) Find the general solution to each of the following differential equations.

a. (4 points) $y'' - 4y' + 5y = 0$

The characteristic equation is $r^2 - 4r + 5 = 0 \Rightarrow$

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i \quad \boxed{\text{2 pts.}}$$

Therefore, $\boxed{\boxed{y = c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x)}}$. $\boxed{\text{2 pts.}}$

b. (6 points) $y^{(3)} + 4y'' + 4y' = 0$

The characteristic equation is $r^3 + 4r^3 + 4r = 0 \Rightarrow r\left(r^2 + 4r + 4\right) = 0 \Rightarrow r(r+2)^2 = 0$.

Thus, the roots of the characteristic equation are 0 and -2 (double root). $\boxed{\text{2 pts.}}$

Therefore, $y = c_1 e^{0x} + c_2 e^{-2x} + c_3 x e^{-2x}$ or $\boxed{\boxed{y = c_1 + c_2 e^{-2x} + c_3 x e^{-2x}}}$. $\boxed{\text{4 pts.}}$

Problem 5. (15 points)

Solve the following initial value problem:

$$y'' - 5y' + 4y = 32x + 20\sin(2x) \quad \text{with } y(0) = 10 \text{ and } y'(0) = 0.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' - 5y' + 4y = 0$.

Characteristic equation: $r^2 - 5r + 4 = 0 \Rightarrow (r-1)(r-4) = 0 \Rightarrow r = 1$ or $r = 4$.

Therefore, $y_c = c_1e^x + c_2e^{4x}$. $\boxed{\text{3 pts.}}$

Step 2. Find y_p using **either** the Method of Undetermined Coefficients **or** the Method of Variation of Parameters.

Method 1. Undetermined Coefficients.

Since the nonhomogeneous term in the d.e. $(32x + 20\sin(2x))$ is the sum of a polynomial of degree 1 and a sine function, we guess that y_p is the sum of a polynomial of degree 1 and the sum of a cosine and sine with the same frequency as the sine function in the nonhomogeneous term: $y_p = Ax + B + C\cos(2x) + D\sin(2x)$. No term in this guess duplicates a term in y_c, so there is no need to modify the guess. $\boxed{\text{3 pts.}}$

$y = Ax + B + C\cos(2x) + D\sin(2x) \Rightarrow y' = A - 2C\sin(2x) + 2D\cos(2x) \Rightarrow$

$y'' = -4C\cos(2x) - 4D\sin(2x)$.

Therefore, the left side of the d.e. is

$y'' - 5y' + 4y = -4C\cos(2x) - 4D\sin(2x) - 5\left[A - 2C\sin(2x) + 2D\cos(2x)\right] + 4\left[Ax + B + C\cos(2x) + D\sin(2x)\right] =$ $4Ax + (4B - 5A) + 10C\sin(2x) - 10D\cos(2x)$. We want this to equal the nonhomogeneous term $32x + 20\sin(2x)$, so $4A = 32,\ 4B - 5A = 0,\ 10C = 20,$ and $-10D = 0 \Rightarrow$

$A = 8,\ B = 10,\ C = 2,$ and $D = 0$. Therefore, $y_p = 8x + 10 + 2\cos(2x)$. $\boxed{\text{6 pts.}}$

Method 2. Variation of Parameters.

From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^x$ and $y_2 = e^{4x}$. The Wronskian is given by $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{4x} \\ e^x & 4e^{4x} \end{vmatrix}$

$= e^x\left(4e^{4x}\right) - (e^x)(e^{4x}) = 4e^{5x} - e^{5x} = 3e^{5x}$. $\boxed{\text{1 pt.}}$

$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{4x}\left(32x + 20\sin(2x)\right)}{3e^{5x}}\,dx = -\frac{32}{3}\int xe^{-x}\,dx - \frac{20}{3}\int e^{-x}\sin(2x)\,dx$

$= -\frac{32}{3}(-x-1)e^{-x} - \frac{4}{3}e^{-x}\left[-\sin(2x) - 2\cos(2x)\right]$ using entries 46 (with $u = -x$) and 49 from the Table of Integrals. $\boxed{\text{3 pts.}}$

$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^x\left(32x + 20\sin(2x)\right)}{3e^{5x}}\,dx = \frac{32}{3}\int xe^{-4x}\,dx + \frac{20}{3}\int e^{-4x}\sin(2x)\,dx$

$= \frac{2}{3}(-4x-1)\,e^{-4x} + \frac{1}{3}e^{-4x}\left[-4\sin(2x) - 2\cos(2x)\right]$ using entries 46 (with $u = -4x$) and 49 from the Table of Integrals. $\boxed{\text{3 pts.}}$

Therefore, $y_p = u_1y_1 + u_2y_2$

$= \left\{-\frac{32}{3}(-x-1)e^{-x} - \frac{4}{3}e^{-x}\left[-\sin(2x) - 2\cos(2x)\right]\right\}e^x$

$+ \left\{\frac{2}{3}(-4x-1)\,e^{-4x} + \frac{1}{3}e^{-4x}\left[-4\sin(2x) - 2\cos(2x)\right]\right\}e^{4x}$

$= \frac{32}{3}(x+1) + \frac{4}{3}\sin(2x) + \frac{8}{3}\cos(2x) - \frac{2}{3}(4x+1) - \frac{4}{3}\sin(2x) - \frac{2}{3}\cos(2x) = 8x + 10 + 2\cos(2x)$ $\boxed{\text{2 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1e^x + c_2e^{4x} + 8x + 10 + 2\cos(2x)$. $\boxed{\text{1 pt.}}$

Step 4. Use the initial conditions to find c_1 and c_2.

$y = c_1e^x + c_2e^{4x} + 8x + 10 + 2\cos(2x) \Rightarrow y' = c_1e^x + 4c_2e^{4x} + 8 - 2\sin(2x)$

$y(0) = 10 \Rightarrow 10 = c_1e^0 + c_2e^0 + 8(0) + 10 + 2\cos(0) = c_1 + c_2 + 12 \Rightarrow c_1 + c_2 = -2$

$y'(0) = 0 \Rightarrow 0 = c_1e^0 + 4c_2e^0 + 8 - 2\sin(0) = c_1 + 4c_2 + 8 \Rightarrow c_1 + 4c_2 = -8$

$c_1 + c_2 = -2,\ c_1 + 4c_2 = -8 \Rightarrow c_1 = 0,\ c_2 = -2$ $\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = -2e^{4x} + 8x + 10 + 2\cos(2x).}}$

Problem 6. (10 points)

A series circuit consists of a generator supplying $120\cos(4t)$ volts, a 5 ohm resistor, and a 0.05 farad capacitor. Assume the initial charge on the capacitor is 0. Find the charge on the capacitor $Q(t)$.

Remember that the d.e. modeling the charge on the capacitor in an RC circuit is $RQ' + \dfrac{Q}{C} = E(t)$

Substituting the given parameter values and the given applied voltage, we find that the model d.e. is $5Q' + \underbrace{\dfrac{1}{0.05}}_{20} Q = 120\cos(4t)$, or $Q' + 4Q = 24\cos(4t)$

This is a first-order linear d.e., so we can solve it by finding the integrating factor or we can treat it as a constant-coefficient linear d.e.:

Step 1. Find Q_c by solving the homogeneous d.e. $Q' + 4Q = 0$.

Characteristic equation: $r + 4 = 0 \Rightarrow r = -4$.

Therefore, $Q_c = ce^{-4t}$. $\boxed{\text{3 pts.}}$

Step 2. Find Q_p using **either** the Method of Undetermined Coefficients **or** the Method of Variation of Parameters. Using the Method of Undetermined Coefficients: Since the nonhomogeneous term in the d.e. $(24\cos(4t))$ is a cosine function, we guess that Q_p is the sum of a cosine and sine with the same frequency as the cosine function in the nonhomogeneous term: $Q_p = A\cos(4t) + B\sin(4t)$. No term in this guess duplicates a term in Q_c, so there is no need to modify the guess.

$Q = A\cos(4t) + B\sin(4t) \Rightarrow Q' = -4A\sin(4t) + 4B\cos(4t)$

Therefore, the left side of the d.e. is $Q' + 4Q = -4A\sin(4t) + 4B\cos(4t) + 4[A\cos(4t) + B\sin(4t)] = [4A + 4B]\cos(4t) + [-4A + 4B]\sin(4t)$. We want this to equal the nonhomogeneous term $24\cos(4t)$, so $4A + 4B = 24$, $-4A + 4B = 0 \Rightarrow A = 3$ and $B = 3$. Therefore, $Q_p = 3\cos(4t) + 3\sin(4t)$. $\boxed{\text{5 pts.}}$

Step 3. $Q = Q_c + Q_p = ce^{-4t} + 3\cos(4t) + 3\sin(4t)$ $\boxed{\text{1 pt.}}$

Step 4. Use the initial condition to find c:

$Q(0) = 0 \Rightarrow 0 = ce^0 + 3\cos(0) + 3\sin(0) = c + 3 \Rightarrow c = -3$ $\boxed{\text{1 pt.}}$ $\Rightarrow$ $\boxed{\boxed{Q = -3e^{-4t} + 3\cos(4t) + 3\sin(4t)}}$

Problem 7. (15 points)

a. (4 pts.) Find $\mathcal{L}\left\{e^{-2t}\cos(3t) - 4t^2\right\}$

Using the Laplace Transform table entries for $e^{at}\cos(kt)$ and t^n, we find that

$$\mathcal{L}\left\{e^{-2t}\cos(3t) - 4t^2\right\} = \mathcal{L}\left\{e^{-2t}\cos(3t)\right\} - 4\mathcal{L}\left\{t^2\right\} = \frac{s-(-2)}{(s-(-2)) + 3^2} - 4\frac{2!}{s^{2+1}} = \boxed{\boxed{= \frac{s+2}{s^2+4s+13} - \frac{8}{s^3}}}$$

$\boxed{\text{4 pts.}}$

b. (5 pts.) Find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} \sin(t) & \text{if } t < 2\pi \\ 0 & \text{if } t \geq 2\pi \end{cases}$$

$f(t) = [1 - u(t-2\pi)]\sin(t) = \sin(t) - u(t-2\pi)\sin(t) = \sin(t) - u(t-2\pi)\sin(t-2\pi)$, $\boxed{\text{2 pts.}}$ so

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t) - u(t-2\pi)\sin(t-2\pi)\} = \frac{1}{s^2+1^2} - e^{-2\pi s}\frac{1}{s^2+1} = \boxed{\boxed{\frac{1-e^{-2\pi s}}{s^2+1}}} \quad \boxed{\text{3 pts.}}$$

c. (6 pts.) Find $\mathcal{L}^{-1}\left\{\dfrac{2s+4}{s^2+4s-5}\right\}$.

Use a partial fraction decomposition: $\dfrac{2s+4}{s^2+4s-5} = \dfrac{2s+4}{(s+5)(s-1)} = \dfrac{A}{s+5} + \dfrac{B}{s-1} \Rightarrow$

$(s+5)(s-1)\left[\dfrac{2s+4}{(s+5)(s-1)}\right] = (s+5)(s-1)\left[\dfrac{A}{s+5} + \dfrac{B}{s-1}\right] \Rightarrow$

$2s+4 = A(s-1) + B(s+5) = (A+B)s + (-A+5B) \Rightarrow A+B=2,\ -A+5B=4 \Rightarrow$

$A=1,\ B=1 \Rightarrow$ $\boxed{\text{4 pts.}}$

$\mathcal{L}^{-1}\left\{\dfrac{2s+4}{s^2+4s-5}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+5} + \dfrac{1}{s-1}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+5}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} = \boxed{\boxed{e^{-5t}+e^t}}$ $\boxed{\text{2 pts.}}$

Problem 8. (15 points)

Use the Laplace Transform to solve the following initial value problem:

$$x'' + 4x' + 8x = 0 \text{ with } x(0) = 0 \text{ and } x'(0) = 2.$$

Solutions to this IVP not using the Laplace transform method will not receive any credit. Primes denote derivatives with respect to t: $x' = dx/dt$ and $x'' = d^2x/dt^2$.

$x'' + 4x' + 8x = 0 \Rightarrow \mathcal{L}\{x'' + 4x' + 8x\} = \mathcal{L}\{0\} \Rightarrow \mathcal{L}\{x''\} + 4\mathcal{L}\{x'\} + 8\mathcal{L}\{x\} = 0$ $\boxed{\text{3 pts.}}$

$\Rightarrow s^2\mathcal{L}\{x\} - sx(0) - x'(0) + 4\left[s\mathcal{L}\{x\} - x(0)\right] + 4\mathcal{L}\{x\} = 0$ $\boxed{\text{3 pts.}}$

$s^2\mathcal{L}\{x\} - s\cdot 0 - 2 + 4\left[s\mathcal{L}\{x\} - 0\right] + 8\mathcal{L}\{x\} = 0 \Rightarrow \left(s^2+4s+8\right)\mathcal{L}\{x\} = 2 \Rightarrow$

$\mathcal{L}\{x\} = \dfrac{2}{s^2+4s+8}$ $\boxed{\text{1 pt.}}$ $\Rightarrow x = \mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4s+8}\right\}$.

The denominator is an irreducible quadratic (complex roots), so we need to complete the square: $s^2+4s+8 = (s+2)^2+4$.

Therefore, $\Rightarrow x = \mathcal{L}^{-1}\left\{\dfrac{2}{(s+2)^2+4}\right\} = \boxed{\boxed{e^{-2t}\sin(2t)}}$ using the table entry for $e^{at}\sin(kt)$. $\boxed{\text{8 pts.}}$