92.236 Engineering Differential Equations Final Exam Solutions
Spring 2014

Problem 1. (5+5 pts.)

The first-order differential equation $xy' = -y$ is separable, linear, exact, and homogeneous. Find its general solution **by using any two different of the above four methods**.

Using the solution procedure for separable first-order d.e.'s: $x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} = -\frac{dx}{x}$ $\boxed{\text{2 pts.}}$

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x} \Rightarrow \ln(y) = -\ln(x) + c_1 \Rightarrow e^{\ln(y)} = e^{-\ln(x)+c_1} = \underbrace{e^{-\ln(x)}}_{=x^{-1}}\underbrace{e^{c_1}}_{=c} \Rightarrow \boxed{\boxed{y = cx^{-1}}}\ \boxed{\text{3 pts.}}$$

Using the solution procedure for linear first-order d.e.'s: First write the equation in standard form.

$xy' = -y \Rightarrow xy' + y = 0 \Rightarrow y' + \left(\frac{1}{x}\right) y = 0$ $\boxed{\text{1 pt.}}$

Next, find the integrating factor: $\rho(x) = e^{\int 1/x\ dx} = e^{\ln(x)} = x.$ $\boxed{\text{1 pt.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x\left[y' + \left(\frac{1}{x}\right) y\right] = x \cdot 0 \Rightarrow xy' + y = 0.$ $\boxed{\text{1 pt.}}$

Use the Product Rule backwards to rewrite the d.e. as $\frac{d}{dx}[xy] = 0.$ $\boxed{\text{1 pt.}}$

Integrating both sides, we obtain $xy = \int 0\ dx = 0 + c.$ $\boxed{\text{1 pt.}}$

Therefore, $xy = c$, so $\boxed{\boxed{y = cx^{-1}.}}$

Using the solution procedure for homogeneous first-order d.e.'s: $xy' = -y \Rightarrow y' = -\frac{y}{x}$. Let $v = y/x$.

Replace y' by $v + xv'$ and replace y by xv: $y' = -\frac{y}{x} \Rightarrow v + xv' = -\frac{xv}{v} = -v$ $\boxed{\text{2 pts.}}$

$$\Rightarrow x\frac{dv}{dx} = -2v \Rightarrow \frac{dv}{v} = -2\frac{dx}{x} \Rightarrow \int \frac{dv}{v} = -2\int \frac{dx}{x} \Rightarrow \ln(v) = -2\ln(x) + c_1 \Rightarrow e^{\ln(v)} = e^{-2\ln(x)+c_1}\ \boxed{\text{2 pts.}}$$

$$= \underbrace{e^{-2\ln(x)}}_{=x^{-2}}\underbrace{e^{c_1}}_{=c} \Rightarrow v = cx^{-2} \Rightarrow \frac{y}{x} = cx^{-2} \Rightarrow \boxed{\text{1 pt.}}\ \boxed{\boxed{y = cx^{-1}}}$$

Using the solution procedure for exact first-order d.e.'s: First write the equation in standard form.

$xy' = -y \Rightarrow \underbrace{y}_{M} + \underbrace{x}_{N} y' = 0$

$\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$ so the d.e. is exact. The solution of the d.e. is $f(x, y) = c$ where f satisfies the conditions $\frac{\partial f}{\partial x} = M = y$ and $\frac{\partial f}{\partial y} = N = x$. $\frac{\partial f}{\partial x} = y \Rightarrow f = \int y\ \partial x = yx + g(y)$ $\boxed{\text{2 pts.}}$

$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[yx + g(y)] = x + g'(y)$. But $\frac{\partial f}{\partial y} = x$ so $x + g'(y) = x \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0.$ $\boxed{\text{2 pts.}}$

Therefore, $f(x, y) = yx$ and the solution of the d.e. is $yx = c \Rightarrow \boxed{\boxed{y = cx^{-1}}}\ \boxed{\text{1 pt.}}$

Problem 2. (15 pts.)

Solve the following initial value problem (IVP) for $x > 0$: $x^2y' - xy = 2y^2$ with $y(1) = -1$.

$x^2y' - xy = 2y^2 \Rightarrow y' = \dfrac{2y^2 + xy}{x^2}$. This is a homogeneous d.e. because y' equals a rational function in which each term has the same degree (2). $\boxed{\text{3 pts.}}$

Let $v = y/x$. Replace y' by $v + xv'$ and replace y by xv:

$$y' = \frac{2y^2 + xy}{x^2} \Rightarrow v + xv' = \frac{2(xv)^2 + x(xv)}{x^2} = \frac{x^2\left(2v^2 + v\right)}{x^2} = 2v^2 + v \Rightarrow x\frac{dv}{dx} = 2v^2 \ \boxed{\text{5 pts.}}$$

$$\Rightarrow \frac{dv}{v^2} = \frac{2\,dx}{x} \Rightarrow \frac{v^{-1}}{-1} = 2\ln(x) + c \Rightarrow -\frac{1}{v} = 2\ln(x) + c \Rightarrow -\frac{x}{y} = 2\ln(x) + c \ \boxed{\text{5 pts.}}$$

$y(1) = -1 \Rightarrow -\dfrac{1}{-1} = 2\ln(1) + c \Rightarrow c = 1$ $\boxed{\text{2 pt.}}$ Therefore

$$-\frac{x}{y} = 2\ln(x) + 1 \Rightarrow \boxed{\boxed{y = -\frac{x}{2\ln(x) + 1}}}$$

Problem 3. (10 pts.) Consider the expansion-contraction equation: $\dfrac{dP}{dt} = P(10 - P)(P - 20)$.

a. Find the critical points of this equation and sketch the phase line. Determine the stability of the critical points and mark the results on the phase line, along with arrows that indicate the sign of the derivative $\dfrac{dP}{dt}$.

$P(10 - P)(P - 20) = 0 \Rightarrow$ $\boxed{\boxed{\text{the critical points are } 0,\ 10, \text{ and } 20}}$ $\boxed{\text{1 pt.}}$

The three critical points divide the phase line into 4 intervals: $P > 20,\ 10 < P < 20,\ 0 < P < 10$ and $P < 0$.

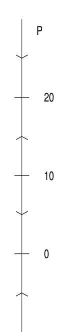
$\left.\dfrac{dP}{dt}\right|_{x=30} = 30(10 - 30)(30 - 20) < 0$, so the direction arrow points down for $P > 20$.

$\left.\dfrac{dP}{dt}\right|_{P=15} = 15(10 - 15)(15 - 20) > 0$, so the direction arrow points up for $10 < P < 20$.

$\left.\dfrac{dP}{dt}\right|_{x=5} = 5(10 - 5)(5 - 20) < 0$, so the direction arrow points down for $0 < P < 10$.

$\left.\dfrac{dP}{dt}\right|_{x=-10} = (-10)(10 - (-10))(-10 - 20) > 0$, so the direction arrow points up for $P < 0$.

$\boxed{\text{4 pts.}}$



From the phase line we can see that $\boxed{\boxed{\text{both 0 and 20 are stable but 10 is unstable}}}$. $\boxed{\text{2 pts.}}$

b. For each of the initial conditions $P(0) = 5$, $P(0) = 15$, and $P(0) = 25$, state where the corresponding particular solutions are headed as $t \to \infty$, that is, find $\lim_{t\to\infty} P(t)$.

Since 5 lies in the interval $0 < P < 10$, we can see from the phase line that $\boxed{\boxed{P(0) = 5 \Rightarrow P(t) \to 0}}$ as t increases. $\boxed{\text{1 pt.}}$

Since 15 lies in the interval $10 < P < 20$, we can see from the phase line that $\boxed{\boxed{P(0) = 15 \Rightarrow P(t) \to 20}}$ as t increases. $\boxed{\text{1 pt.}}$

Since 25 lies in the interval $P > 20$, we can see from the phase line that $\boxed{\boxed{P(0) = 25 \Rightarrow P(t) \to 20}}$ as t increases. $\boxed{\text{1 pt.}}$

Problem 4. (15 points)

A 20-liter tank initially contains 10 liters of water in which 5 grams of salt are dissolved. A solution containing 20 grams of salt per liter is pumped into the tank at the rate of 2 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 1 liter per minute. How much salt will the tank contain when it is full?

Let t denote time (in minutes) and let x denote the amount of salt in the tank at time t.

$\frac{dx}{dt}$ = rate in - rate out $\boxed{\text{1 pt.}}$

= (flow rate in)(concentration in) - (flow rate out)(concentration out), so

$\frac{dx}{dt} = \left(2\frac{\text{liters}}{\text{minute}}\right)\left(20\frac{\text{g}}{\text{liter}}\right) - \left(1\frac{\text{liters}}{\text{minute}}\right)\left(\frac{x\text{ g}}{(10+t)\text{ liters}}\right)$. $\boxed{\text{5 pts.}}$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 10 + (2-1)t$ liters.)

Initially there are 5 grams of salt in the tank, so $x(0) = 5$

Therefore, the initial value problem describing this mixing problem is $\frac{dx}{dt} = 40 - \frac{x}{10+t}$ with $x(0) = 5$.

The d.e. is linear. First write the equation in standard form.

$\frac{dx}{dt} = 40 - \frac{x}{10+t} \Rightarrow \frac{dx}{dt} + \left(\frac{1}{10+t}\right)x = 40 \Rightarrow$ $\boxed{\text{1 pt.}}$

Next, find the integrating factor: $\rho(t) = e^{\int 1/(10+t)\ dt} = e^{\ln(10+t)} = 10 + t$. $\boxed{\text{3 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$(10+t)\left[\frac{dx}{dt} + \left(\frac{1}{10+t}\right)x\right] = 40(10+t)$. $\boxed{\text{1 pt.}}$

Use the Product Rule backwards to rewrite the d.e. as $\frac{d}{dt}[(10+t)x] = 40(10+t)$. $\boxed{\text{1 pt.}}$

Integrating both sides, we obtain $(10+t)x = \int 40(10+t)\ dt = 20(10+t)^2 + c$. $\boxed{\text{1 pt.}}$

$x(0) = 5 \Rightarrow (10+0)5 = 20(10+0)^2 + c \Rightarrow c = -1950$ $\boxed{\text{1 pt.}}$ so $(10+t)x = 20(10+t)^2 - 1950$

The tank is full at time $t = 10$ minutes. At $t = 10$ we have $(10+10)x = 20(10+10)^2 - 1950 \Rightarrow$

$\boxed{\boxed{x(10) = 302.5 \text{ grams}}}$ $\boxed{\text{1 pt.}}$

Problem 5. (5+5 pts.) Find the general solution to each of the following third-order linear homogeneous differential equations:

a. $y''' - 8y'' + 16y' = 0$

The characteristic equation is $r^3 - 8r^2 + 16r = 0 \Rightarrow r\left(r^2 - 8r + 16\right) = 0 \Rightarrow r(r-4)^2 = 0 \Rightarrow$

$r = 0$ or $r = 4$ (double root) $\boxed{\text{2 pts.}}$

Therefore, $y = c_1 e^{0x} + c_2 e^{4x} + c_3 x e^{4x}$, or $\boxed{\boxed{y = c_1 + c_2 e^{4x} + c_3 x e^{4x}}}$. $\boxed{\text{3 pts.}}$

b. $y''' - y' = 0$

The characteristic equation is $r^3 - r = 0 \Rightarrow r\left(r^2 - 1\right) = 0 \Rightarrow r(r+1)(r-1) = 0 \Rightarrow$
$r = 0,\ r = -1,$ or $r = 1$ $\boxed{\text{2 pts.}}$

Therefore, $y = c_1 e^{0x} + c_2 e^{-1 \cdot x} + c_3 e^{1x}$ or $\boxed{\boxed{y = c_1 + c_2 e^{-x} + c_3 e^{x}}}$. $\boxed{\text{3 pts.}}$

Problem 6. (15 points)

Consider a forced, undamped mass-spring system with mass $m = 1$ kg, spring constant $k = 9$ N/m, and an external force $F_{\text{ext}}(t) = 10\cos(2t)$ N. Find the position function $x(t)$ for the initial conditions $x(0) = 2$ and $x'(0) = 3$.

The d.e. modeling a mass-spring system is $mx'' + cx' + kx = F_{\text{ext}}(t)$, where m denotes the mass of the object, c denotes the damping coefficient, k denotes the spring constant, and $F_{\text{ext}}(t)$ denotes the external force. Substituting the given parameter values, we obtain the equation $x'' + 9x = 10\cos(2t)$ $\boxed{\text{3 pts.}}$

Step 1. Find x_c by solving the homogeneous d.e. $x'' + 9x' = 0$.

Characteristic equation: $r^2 + 9 = 0 \Rightarrow r^2 = -9 \Rightarrow r = \sqrt{-9} = \pm 3i$.

Therefore, $x_c = c_1 \cos(3t) + c_2 \sin(3t)$. $\boxed{\text{3 pts.}}$

Step 2. Find x_p using **either** the Method of Undetermined Coefficients **or** the Method of Variation of Parameters.

Method 1. Undetermined Coefficients.

Since the nonhomogeneous term in the d.e. $(10\cos(2t))$ is a cosine function, we guess that x_p is the sum a cosine and sine with the same frequency as the cosine function in the nonhomogeneous term: $x_p = A\cos(2t) + B\sin(2t)$. No term in this guess duplicates a term in x_c, so there is no need to modify the guess. $\boxed{\text{2 pts.}}$

$x = A\cos(2t) + B\sin(2t) \Rightarrow x' = -2A\sin(2t) + 2B\cos(2t) \Rightarrow x'' = -4A\cos(2t) - 4B\sin(2t)$.

Therefore, the left side of the d.e. is
$x'' + 9x = -4A\cos(2t) - 4B\sin(2t) + 9\left[A\cos(2t) + B\sin(2t)\right] = 5A\cos(2t) + 5B\sin(2t)$. We want this to equal the nonhomogeneous term $10\cos(2t)$, so $5A = 10,$ and $5B = 0 \Rightarrow A = 2$ and $B = 0$. Therefore, $x_p = 2\cos(2t)$. $\boxed{\text{4 pts.}}$

Method 2. Variation of Parameters.

From x_c we obtain two independent solutions of the homogeneous d.e: $x_1 = \cos(3t)$ and
$x_2 = \sin(3t)$. The Wronskian is given by $W(x) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix}$
$= \cos(3t)\left(3\cos(3t)\right) - (-3\sin(3t))(\sin(3t)) = 3\cos^2(3t) + 3\sin^2(3t) = 3\left[\cos^2(3t) + \sin^2(3t)\right] = 3.$ $\boxed{\text{1 pt.}}$

$$u_1 = \int \frac{-x_2 F_{\text{ext}}(t)}{W(t)}\, dx = -\int \frac{\sin(3t)\left(10\cos(2t)\right)}{3}\, dx = -\frac{10}{3}\int \sin(3t)\cos(2t)\ dt$$

$= -\dfrac{10}{3}\left[-\dfrac{\cos(t)}{2} - \dfrac{\cos(5t)}{10}\right] = \dfrac{5}{3}\cos(t) + \dfrac{1}{3}\cos(5t)$ using entry 31 from the Table of Integrals. $\boxed{\text{2 pts.}}$

$$u_2 = \int \frac{x_1 F_{\text{ext}}(t)}{W(t)}\, dx = \int \frac{\cos(3t)\left(10\cos(2t)\right)}{3}\, dx = \frac{10}{3}\int \cos(3t)\cos(2t)\ dt$$

$= \dfrac{10}{3}\left[\dfrac{\sin(t)}{2} + \dfrac{\sin(5t)}{10}\right] = \dfrac{5}{3}\sin(t) + \dfrac{1}{3}\sin(5t)$ using entry 30 from the Table of Integrals. $\boxed{2 \text{ pts.}}$

Therefore, $x_p = u_1x_1 + u_2x_2 = \left[\dfrac{5}{3}\cos(t) + \dfrac{1}{3}\cos(5t)\right]\cos(3t) + \left[\dfrac{5}{3}\sin(t) + \dfrac{1}{3}\sin(5t)\right]\sin(3t)$

$= \dfrac{5}{3}\left[\cos(t)\cos(3t) + \sin(t)\sin(3t)\right] + \dfrac{1}{3}\left[\cos(5t)\cos(3t) + \sin(5t)\sin(3t)\right] = \dfrac{5}{3}\cos(t-3t) + \dfrac{1}{3}\cos(5t-3t)$

$= \dfrac{5}{3}\cos(2t) + \dfrac{1}{3}\cos(2t) = 2\cos(2t)$ $\boxed{1 \text{ pt.}}$

Step 3. $x = x_c + x_p$, so $x = c_1\cos(3t) + c_2\sin(3t) + 2\cos(2t)$. $\boxed{1 \text{ pt.}}$

Step 4. Use the initial conditions to find c_1 and c_2.

$x = c_1\cos(3t) + c_2\sin(3t) + 2\cos(2t) \Rightarrow x' = -3c_1\sin(3t) + 3c_2\cos(3t) - 4\sin(2t)$

$x(0) = 2 \Rightarrow 2 = c_1\cos(0) + c_2\sin(0) + 2\cos(0) = c_1 + 2 \Rightarrow c_1 = 0$
$x'(0) = 3 \Rightarrow 3 = -3c_1\sin(0) + 3c_2\cos(0) - 4\sin(0) = 3c_2 \Rightarrow c_2 = 1$ $\boxed{2 \text{ pts.}}$

Therefore, $\boxed{\boxed{y = \sin(3t) + 2\cos(2t).}}$

Problem 7. (5+5 points) Find the inverse Laplace transforms of the following two functions:

a. $\dfrac{s+2}{s^2+4}$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \boxed{\boxed{\cos(2t) + \sin(2t)}}$$

using the Laplace Transform table entries for $\sin(kt)$ and $\cos(kt)$ with $k = 2$.

b. Find $\dfrac{s}{(s+2)^2+4}$.

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+4}\right\}$$

$\boxed{\boxed{= e^{-2t}\cos(2t) - e^{-2t}\sin(2t)}}$ using the Laplace Transform table entries for $e^{at}\sin(kt)$ and $e^{at}\cos(kt)$ with $a = -2$ and $k = 2$.

Problem 8. (15 points)

Use the Laplace Transform to solve the following IVP: $x'' - x' = 1; x(0) = 2; x'(0) = 1$

Solutions not using the Laplace transform method will not receive any credit.

$x'' - x' = 1 \Rightarrow \mathcal{L}\{x'' - x'\} = \mathcal{L}\{1\} \Rightarrow \mathcal{L}\{x''\} - \mathcal{L}\{x'\} = \dfrac{1}{s}$ $\boxed{3 \text{ pts.}}$

$\Rightarrow \left[s^2\mathcal{L}\{x\} - sx(0) - x'(0)\right] - \left[s\mathcal{L}\{x\} - x(0)\right] = \dfrac{1}{s}$ $\boxed{3 \text{ pts.}}$

$\left[s^2\mathcal{L}\{x\} - s\cdot 2 - 1\right] - \left[s\mathcal{L}\{x\} - 2\right] = \dfrac{1}{s} \Rightarrow \left(s^2 - s\right)\mathcal{L}\{x\} = \dfrac{1}{s} + 2s - 1 = \dfrac{1 + 2s^2 - s}{s} = \dfrac{2s^2 - s + 1}{s} \Rightarrow$

$\mathcal{L}\{x\} = \dfrac{2s^2 - s + 1}{s\left(s^2 - s\right)} = \dfrac{2s^2 - s + 1}{s^2(s-1)}$ $\boxed{1 \text{ pt.}}$ $\Rightarrow x = \mathcal{L}^{-1}\left\{\dfrac{2s^2 - s + 1}{s^2(s-1)}\right\}$.

Use a partial fraction decomposition: $\dfrac{2s^2 - s + 1}{s^2(s-1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1}$

$$s^2(s-1)\left[\frac{2s^2-s+1}{s^2(s-1)}\right] = s^2(s-1)\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}\right] \Rightarrow$$

$2s^2 - s + 1 = As(s-1) + B(s-1) + Cs^2 = As^2 - As + Bs - B + Cs^2 = (A+C)s^2 + (-A+B)s - B$

$\Rightarrow A + C = 2,\ -A + B = -1,\ -B = 1 \Rightarrow A = 0,\ B = -1,\ C = 2.$ $\boxed{\text{6 pts.}}$

Therefore, $x = \mathcal{L}^{-1}\left\{\frac{0}{s}+\frac{-1}{s^2}+\frac{2}{s-1}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \Rightarrow \boxed{\boxed{x = -t + 2e^t}}$ $\boxed{\text{2 pts.}}$