92.236 Engineering Differential Equations Final Exam Solutions
Spring 2015

Problem 1. (15 pts.)

Solve the following initial value problem: $y' = \dfrac{2y + x^2}{x}$, $y(1) = 2$.

This is a linear d.e. $\boxed{\text{3 pts.}}$ Put the equation into standard form for a linear equation:

$$y' = \frac{2y + x^2}{x} = \frac{2y}{x} + \frac{x^2}{2} \Rightarrow y' - \left(\frac{2}{x}\right) y = x \ \boxed{\text{1 pt.}}$$

Find the integrating factor: $\rho(x) = e^{\int -2/x \ dx} = e^{-2\ln(x)} = x^{-2}$. $\boxed{\text{4 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^{-2}\left[y' - 2y/x\right] = x^{-2} \cdot x = x^{-1}$. $\boxed{\text{1 pt.}}$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\left[x^{-2}y\right] = x^{-1}$. $\boxed{\text{3 pts.}}$

Integrating both sides, we obtain $x^{-2}y = \displaystyle\int x^{-1} \ dx = \ln(x) + c$. $\boxed{\text{2 pts.}}$

$y(1) = 2 \Rightarrow 1^{-2}(2) = \ln(1) + c \Rightarrow c = 2$. $\boxed{\text{1 pt.}}$

Therefore, $x^{-2}y = \ln(x) + 2$, so $\boxed{\boxed{y = x^2\ln(x) + 2x^2.}}$

Problem 2. (10 pts.)

Solve the following initial value problem: $y' = \dfrac{xy + y^2}{x^2}$, $y(1) = -1$.

Since y' equals a rational function in which each term has the same degree (2), the d.e. is homogeneous $\boxed{\text{2 pts.}}$.

We introduce the new variable $v = y/x$. In the d.e. we replace y' by $v + x\dfrac{dv}{dx}$ and we replace y by xv:

$$y' = \frac{xy + y^2}{x^2} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{x(xv) + (xv)^2}{x^2}}_{\boxed{\text{3 pts.}}} = \frac{x^2\left(v + v^2\right)}{x^2} = v + v^2 \Rightarrow \underbrace{x\frac{dv}{dx} = v^2}_{\boxed{\text{1 pt.}}}$$

$$\Rightarrow \frac{dv}{v^2} = \frac{dx}{x} \Rightarrow \int v^{-2} \ dv = \int \frac{1}{x} \ dx \Rightarrow \underbrace{\frac{v^{-1}}{-1} = \ln(x) + c}_{\boxed{\text{2 pts.}}} \Rightarrow \underbrace{-\left(y/x\right)^{-1} = \ln(x) + c}_{\boxed{\text{1 pts.}}}$$

$$y(1) = -1 \Rightarrow -\left(\frac{-1}{1}\right)^{-1} = \ln(1) + c \Rightarrow c = 1 \Rightarrow \boxed{\text{1 pt.}}$$

$$-\left(y/x\right)^{-1} = \ln(x) + 1 \Rightarrow (y/x) = -\frac{1}{\ln(x) + 1} \Rightarrow \boxed{\boxed{y = -\frac{x}{\ln(x) + 1}}}.$$

Problem 3. (10 pts.)

A tank initially contains 50 liters of water in which 250 grams of salt are dissolved. Water containing 10 grams per liter of salt in pumped into the tank at the rate of 5 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 5 liters per minute. How many grams of salt will be in the tank after 10 minutes?

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in grams).

$\frac{dx}{dt} = \text{rate in - rate out}$

$= \text{(flow rate in)(concentration in) - (flow rate out)(concentration out), so}$

$\frac{dx}{dt} = \left(5\frac{\text{liters}}{\text{minute}}\right)\left(10\frac{\text{gm}}{\text{liter}}\right) - \left(5\frac{\text{liters}}{\text{minute}}\right)\left(\frac{x\text{ gm}}{50\text{ liters}}\right).$

(The volume in the tank at time t is initial volume $+\ t$ (flow rate in - flow rate out) $= 50 + (5-5)t$ liters.)

Initially there are 250 gm. of salt in the tank, so $x(0) = 250$

Therefore, the initial value problem describing this mixing problem is

$\frac{dx}{dt} = 50 - \frac{5x}{50} = 50 - \frac{x}{10}$ with $x(0) = 250$. $\boxed{\text{4 pts.}}$

The d.e. is both separable and linear. Treating it as a separable equation, we have

$\frac{dx}{dt} = 50 - \frac{x}{10} = -\frac{1}{10}(x-500) \Rightarrow \frac{dx}{x-500} = -\frac{1}{10}\,dt \Rightarrow \int \frac{dx}{x-500} = -\frac{1}{10}\,dt \Rightarrow \ln|x-500| = -t/10 + c$

$\Rightarrow |x-500| = e^{-t/10+c} = e^{-t/10}e^{c} \Rightarrow x - 500 = \underbrace{\pm e^{c}}_{c_1} e^{-t/10}$ $\boxed{\text{4 pts.}}$

$x(0) = 250 \Rightarrow 250 - 500 = c_1 e^0 \Rightarrow c_1 = -250$ $\boxed{\text{1 pt.}}$

$\Rightarrow x - 500 = -250e^{-t/10} \Rightarrow x = 500 - 250e^{-t/10}$

Therefore, $\boxed{\boxed{x(10) = 500 - 250e^{-10/10} \approx 408 \text{ grams.}}}$ $\boxed{\text{1 pt.}}$

Problem 4. (10 pts.) Find the general solution to each of the following linear homogeneous differential equations:

a. (5 pts.) $y''' + 4y' = 0$

The characteristic equation is $r^3 + 4r = 0 \Rightarrow r\left(r^2+4\right) = 0 \Rightarrow$

$r = 0$ or $r = \pm 2i = 0 \pm 2i$ $\boxed{\text{2 pts.}}$

Therefore, $y = c_1e^{0x} + c_2e^{0x}\cos(2x) + c_3e^{0x}\sin(2x)$, or $\boxed{\boxed{y = c_1 + c_2\cos(2x) + c_3\sin(2x)}}$. $\boxed{\text{3 pts.}}$

b. (5 pts.) $y^{(4)} + 2y''' + 5y'' = 0$

The characteristic equation is $r^4 + 2r^3 + 5r^2 = 0 \Rightarrow r^2\left(r^2 + 2r + 5\right) = 0 \Rightarrow$

$r = 0$ (double root) or $r = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$ $\boxed{\text{2 pts.}}$

Therefore, $y = c_1e^{0x} + c_2xe^{0x} + c_3e^{-x}\cos(2x) + c_4e^{-x}\sin(2x)$, or

$\boxed{\boxed{y = c_1 + c_2x + c_3e^{-x}\cos(2x) + c_4e^{-x}\sin(2x)}}$. $\boxed{\text{3 pts.}}$

Problem 5. (15 pts.)

Solve the following initial value problem: $y'' - y = 4e^x,\ y(0) = 1,\ y'(0) = 1.$

Step 1. Find y_c by solving the d.e. $y'' - y = 0$.
Characteristic equation: $r^2 - 1 = 0 \Rightarrow (r+1)(r-1) = 0 \Rightarrow r = -1$ or $r = 1$.
Therefore, $y_c = c_1 e^{-x} + c_2 e^x$. $\boxed{\text{3 pts.}}$

Step 2. Find y_p. You can use either of the following methods.

Method 1: Undetermined Coefficients. The nonhomogeneous term in the de is $4e^x$, an exponential function. We should therefore guess that y_p is an exponential function with the same coefficient of x: $y_p = Ae^x$. This duplicates the term $c_2 e^x$ in y_c, so we need to multiply by x: $y_p = Axe^x$. $\boxed{\text{4 pts.}}$

$y = Axe^x \Rightarrow y' = A(e^x + xe^x)$ (Product Rule) $\Rightarrow y'' = A[e^x + (e^x + xe^x)] = A(2e^x + xe^x)$
Therefore, the left side of the d.e. is $y'' - y = A(2e^x + xe^x) - Axe^x = 2Ae^x$.
We want this to equal the nonhomogeneous term $4e^x$:
$2Ae^x = 4e^x \Rightarrow 2A = 4 \Rightarrow A = 2$ Thus, $y_p = 2xe^x$. $\boxed{\text{5 pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-x}$ and $y_2 = e^x$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = e^{-x}(e^x) - (-e^{-x})(e^x) = 2. \ \boxed{\text{1 pt.}}$$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^x(4e^x)}{2}\,dx = -2\int e^{2x}\,dx = -e^{2x}. \ \boxed{\text{3 pts.}}$$

$$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-x}(4e^x)}{2}\,dx = 2\int dx = 2x. \ \boxed{\text{3 pts.}}$$

Therefore, $y_p = u_1 y_1 + u_2 y_2 = \left[-e^{2x}\right]e^{-x} + [2x]e^x = -e^x + 2xe^x$ $\boxed{\text{1 pt.}}$

Step 3. $y = y_c + y_p$, so $y = c_1 e^{-x} + c_2 e^x + 2xe^x$ $\boxed{\text{1 pt.}}$ (If you used the Method of Variation of Parameters, you can combine the $-e^x$ term in your y_p with the $c_2 e^x$ term in y_c.)

Step 4. Use the initial conditions to determine the values of c_1 and c_2.
$y = c_1 e^{-x} + c_2 e^x + 2xe^x \Rightarrow y' = -c_1 e^{-x} + c_2 e^x + 2e^x + 2xe^x$
$y(0) = 1 \Rightarrow c_1 e^0 + c_2 e^0 + 2(0)e^0 = 1 \Rightarrow c_1 + c_2 = 1$.
$y'(0) = 1 \Rightarrow -c_1 e^0 + c_2 e^0 + 2e^0 + 2(0)e^0 = 1 \Rightarrow -c_1 + c_2 = -1$.

$c_1 + c_2 = 1,\ -c_1 + c_2 = -1 \Rightarrow c_1 = 1,\ c_2 = 0$ $\boxed{\text{2 pts.}}$ Therefore, $\boxed{y = e^{-x} + 2xe^x}$

Problem 6. (15 points)

Consider a forced, undamped mass-spring system with mass 1 kg, damping coefficient 0 Ns/m, spring constant 9 N/m, and an external force $F_{\text{ext}}(t) = 10\cos(2t)$ N. Suppose the initial position and velocity of the mass are given by $x(0) = 2$ m and $x'(0) = 3$ m/s. Find the position function $x(t)$.

The d.e. modeling this system is $mx'' + cx' + kx = F_e(t)$, or $x'' + 0x' + 9x = 10\cos(2t)$. $\boxed{\text{2 pts.}}$

Step 1. Find x_c by solving the d.e. $x'' + 9x = 0$.
Characteristic equation: $r^2 + 9 = 0 \Rightarrow r^2 = -9 \Rightarrow r = \pm\sqrt{-9} = 0 \pm 3i$
Therefore, $x_c = c_1 e^{0t}\cos(3t) + c_2 e^{0t}\sin(3t) = c_1\cos(3t) + c_2\sin(3t)$. $\boxed{\text{3 pts.}}$

Step 2. Find y_p. You can use either of the following methods.

Method 1: Undetermined Coefficients. The nonhomogeneous term in the de is $10\cos(2t)$, a cosine function. We should therefore guess that x_p is a combination of a cosine function and a sine function with the same coefficient of t: $x_p = A\cos(2t) + B\sin(2t)$. This guess does not duplicate any term x_c, so there is no need to modify this guess. $\boxed{\text{3 pts.}}$

$x = A\cos(2t) + B\sin(2t) \Rightarrow x' = -2A\sin(2t) + 2B\cos(2t) \Rightarrow x'' = -4A\cos(2t) - 4B\sin(2t)$
Therefore, the left side of the d.e. is $x''+9x = -4A\cos(2t) - 4B\sin(2t) + 9[A\cos(2t) + B\sin(2t)] = 5A\cos(2t) + 5B\sin(2t)$.
We want this to equal the nonhomogeneous term $10\cos(2t)$:
$5A\cos(2t) + 5B\sin(2t) = 10\cos(2t) \Rightarrow 5A = 10,\ 5B = 0 \Rightarrow A = 2,\ B = 0$. Thus, $x_p = 2\cos(2t)$.
$\boxed{\text{4 pts.}}$

Method 2: Variation of Parameters. From x_c we obtain two independent solutions of the homogeneous d.e: $x_1 = \cos(3t)$ and $x_2 = \sin(3t)$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$$W(x) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = \cos(3t)\,(3\cos(3t)) - (-3\sin(3t))\,(\sin(3t)) =$$

$3\cos^2(3t) + 3\sin^2(3t) = 3$. $\boxed{\text{1 pt.}}$

$$u_1 = \int \frac{-x_2 F_{\text{ext}}(t)}{W(t)}\,dt = -\int \frac{\sin(3t)\,(10\cos(2t))}{3}\,dt = -\frac{10}{3}\int \sin(3t)\cos(2t)\,dt =$$

$-\frac{10}{3}\left[-\frac{\cos((3-2)t)}{2(3-2)} - \frac{\cos((3+2)t)}{2(3+2)}\right] = \frac{5}{3}\cos(t) + \frac{1}{3}\cos(5t)$ using formula 31 from the integral table. $\boxed{\text{2 pts.}}$

$$u_2 = \int \frac{x_1 F_{\text{ext}}(t)}{W(t)}\,dt = \int \frac{\cos(3t)\,(10\cos(2t))}{3}\,dt = \frac{10}{3}\int \cos(3t)\cos(2t)\,dt =$$

$\frac{10}{3}\left[\frac{\sin((3-2)t)}{2(3-2)} + \frac{\sin((3+2)t)}{2(3+2)}\right] = \frac{5}{3}\sin(t) + \frac{1}{3}\sin(5t)$ using formula 30 from the integral table.
$\boxed{\text{2 pts.}}$

Therefore, $x_p = u_1x_1 + u_2x_2 = \left[\frac{5}{3}\cos(t) + \frac{1}{3}\cos(5t)\right]\cos(3t) + \left[\frac{5}{3}\sin(t) + \frac{1}{3}\sin(5t)\right]\sin(3t) =$

$$\frac{5}{3}\underbrace{[\cos(3t)\cos(t) + \sin(3t)\sin(t)]}_{=\cos(2t)} + \frac{1}{3}\underbrace{[\cos(5t)\cos(3t) + \sin(5t)\sin(3t)]}_{=\cos(2t)} = \frac{5}{3}\cos(2t) + \frac{1}{3}\cos(2t) = 2\cos(2t)$$

$\boxed{\text{1 pt.}}$

Step 3. $x = x_c + x_p$, so $x = c_1\cos(3t) + c_2\sin(3t) + 2\cos(2t)$ $\boxed{\text{1 pt.}}$

Step 4. Use the initial conditions to determine the values of c_1 and c_2.
$x = c_1\cos(3t) + c_2\sin(3t) + 2\cos(2t) \Rightarrow x' = -3c_1\sin(3t) + 3c_2\cos(3t) - 4\sin(2t)$
$x(0) = 2 \Rightarrow c_1\cos(0) + c_2\sin(0) + 2\cos(0) = 2 \Rightarrow c_1 + 2 = 2 \Rightarrow c_1 = 0$.
$x'(0) = 3 \Rightarrow -3c_1\sin(0) + 3c_2\cos(0) + 4\sin(0) = 3 \Rightarrow 3c_2 = 3 \Rightarrow c_2 = 1$. $\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = \sin(3t) + 2\cos(2t)}}$

Problem 7. (10 points)

a. (3 pts.) Find the Laplace transform of te^{2t}

Using the Laplace transform table entry for $\mathcal{L}\left\{t^n e^{at}\right\}$ we have $\mathcal{L}\left\{te^{2t}\right\} = \frac{1!}{(s-2)^{1+1}} \boxed{\boxed{= \frac{1}{(s-2)^2}}}$

$\boxed{\text{3 pts.}}$

b. (7 pts.) Find the inverse Laplace transform of $\dfrac{2}{s^3+s}$.

Use a partial fraction decomposition: $\dfrac{2}{s^3+s} = \dfrac{2}{s\left(s^2+1\right)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1}$ $\boxed{\text{2 pts.}}$

$s\left(s^2+1\right)\left[\dfrac{2}{s\left(s^2+1\right)}\right] = s\left(s^2+1\right)\left[\dfrac{A}{s} + \dfrac{Bs+C}{s^2+1}\right] \Rightarrow$

$2 = A\left(s^2+1\right) + (Bs+C)s = (A+B)s^2 + Cs + A \Rightarrow$

$A+B=0,\ C=0,\ A=2 \Rightarrow A=2,\ B=-2,\ C=0$ $\boxed{\text{3 pts.}}$

Therefore, $\mathcal{L}^{-1}\left\{\dfrac{2}{s^3+s}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2}{s} + \dfrac{-2s}{s^2+1}\right\} = 2\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\} = 2(1) - 2\cos(1t)$

$\boxed{\boxed{= 2 - 2\cos(t)}}$ $\boxed{\text{2 pts.}}$

Problem 8. (15 points)

Use the Laplace Transform to solve the following IVP: $x'' - 4x = 8t,\ x(0) = 1,\ x'(0) = 0$.

Solutions not using the Laplace transform method will not receive any credit. x is a function of t.
x'' means $\dfrac{d^2x}{dt^2}$.

$x'' - 4x = 8t \Rightarrow \mathcal{L}\left\{x'' - 4x\right\} = \mathcal{L}\left\{8t\right\} \Rightarrow \mathcal{L}\left\{x''\right\} - 4\mathcal{L}\{x\} = 8\mathcal{L}\left\{t\right\} = \dfrac{8}{s^2}$ $\boxed{\text{3 pts.}}$

$\Rightarrow \left[s^2\mathcal{L}\{x\} - sx(0) - x'(0)\right] - 4\mathcal{L}\{x\} = \dfrac{8}{s^2}$ $\boxed{\text{3 pts.}}$

$\left[s^2\mathcal{L}\{x\} - s\cdot 1 - 0\right] - 4\mathcal{L}\{x\} = \dfrac{8}{s^2} \Rightarrow \left(s^2-4\right)\mathcal{L}\{x\} = \dfrac{8}{s^2} + s = \dfrac{8+s^3}{s^2} \Rightarrow$

$\mathcal{L}\{x\} = \dfrac{s^3+8}{s^2\left(s^2-4\right)}$ $\boxed{\text{1 pt.}}$ $\Rightarrow x = \mathcal{L}^{-1}\left\{\dfrac{s^3+8}{s^2\left(s^2-4\right)}\right\}$.

Use a partial fraction decomposition:

$\dfrac{s^3+8}{s^2\left(s^2-4\right)} = \dfrac{s^3+8}{s^2(s+2)(s-2)} = \dfrac{As+B}{s^2} + \dfrac{C}{s+2} + \dfrac{D}{s-2}$

$s^2(s+2)(s-2)\left[\dfrac{s^3+8}{s^2(s+2)(s-2)}\right] = s^2(s+2)(s-2)\left[\dfrac{As+B}{s^2} + \dfrac{C}{s+2} + \dfrac{D}{s-2}\right] \Rightarrow$

$s^3 + 8 = (As+B)(s+2)(s-2) + Cs^2(s-2) + Ds^2(s+2) = (As+B)(s^2-4) + Cs^2(s-2) + Ds^2(s+2) =$
$(A+C+D)s^3 + (B-2C+2D)s^2 - 4As - 4B$

$\Rightarrow A+C+D=1,\ B-2C+2D=0,\ -4A=0,\ -4B=8 \Rightarrow A=0,\ B=-2,\ C=0,\ D=1.$ $\boxed{\text{6 pts.}}$

Therefore, $x = \mathcal{L}^{-1}\left\{\dfrac{-2}{s^2} + \dfrac{1}{s-2}\right\} = -2\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{s-2}\right\} \Rightarrow$ $\boxed{\boxed{x = -2t + e^{2t}}}$ $\boxed{\text{2 pts.}}$