

Necessary Skills

I. First Order D.E.'s

A. Analytical Techniques. You should be able to recognize and solve the following types of first order d.e.'s:

- separable
- linear
- homogeneous
- exact

B. Qualitative Techniques. You should be able to

- be able to sketch solution curves of a first-order d.e. given the slope field
- find the critical points of an autonomous first-order d.e.
- draw the phase line and solution curves of an autonomous first-order d.e.
- determine the stability of critical points of an autonomous first-order d.e.
- determine the long-term behavior of solutions of an autonomous first-order d.e. using the phase line

C. Applications. You should be able to formulate and solve first order d.e.'s to analyze the following types of problem:

- radioactive decay
- compound interest
- cooling/heating
- mixture
- population models
- 1D motion of an object (given information about the forces on the object or the object's acceleration)

II. Higher Order Linear D.E.'s

A. Analytical Techniques. You should

- know that the general solution of an n^{th} order linear homogeneous d.e. has the form $y = c_1y_1 + c_2y_2 + \dots + c_ny_n$, where y_1, y_2, \dots, y_n are independent solutions of the d.e.
- know that the general solution of an n^{th} order linear nonhomogeneous d.e. has the form $y = y_c + y_p$ where y_c is the general solution of the corresponding homogeneous d.e. and y_p is a particular solution of the given nonhomogeneous d.e.
- be able to solve n^{th} order linear homogeneous d.e.'s with constant coefficients
- be able to find a particular solution of a nonhomogeneous linear equation using either the Method of Undetermined Coefficients or the Method of Variation of Parameters
- be able to find the values of the arbitrary constants in the general solution of an n^{th} order d.e., given n initial conditions

B. Applications.

1. Mass-spring systems. You should

- be able to formulate and solve the second-order linear homogeneous d.e. describing the motion (forced or unforced, damped or undamped) of a mass attached to a spring:
 $mx'' + cx' + kx = F(t)$
- be able to rewrite the expression $c_1 \cos(\omega t) + c_2 \sin(\omega t)$ in the form $C \cos(\omega t - \alpha)$
- be able to tell whether a system is overdamped, underdamped, or critically damped
- be able to find the steady-state periodic solution and the transient solution of a damped, periodically forced mass-spring system
- be able to find the period and frequency of a sinusoidal function

2. LRC circuits. You should

- be able to formulate and solve the second-order linear nonhomogeneous d.e. describing the forced motion of an LRC circuit:
 $LQ'' + RQ' + \frac{1}{C}Q = E(t)$
- be able to find the steady-state periodic solution and the transient solution of an LRC circuit.

III. Systems of First Order D.E.'s. You should be able to

- transform an n^{th} order d.e. into a system of n first order d.e.'s
- solve systems of 2 linear constant coefficient d.e.'s
- formulate systems of equations describing multi-tank mixture problems and coupled mass-spring systems

IV. Laplace Transforms. You should be able to

- find the Laplace transform of a given function using the definition
- find the Laplace transform of a given function using the tables
- find the inverse Laplace transform of a given function using the tables, partial fraction decomposition, and/or completing the square
- solve initial value problems using the Laplace transform method

V. General. You should be able to

- determine the order of a given d.e., and you should know that the number of arbitrary constants in the general solution of a d.e. equals the order
- determine whether a given function is a solution of a given d.e.
- translate a verbal description of a physical system into a d.e.

There is no guarantee that the actual exam will bear any resemblance to this practice exam. The purpose of the practice exam is to give you an idea of the approximate length and the type of problem that you can expect on the actual exam.

Problem 1. (15 pts.)

Solve the following initial value problem: $y' + y = x$, $y(0) = 1$.

Problem 2. (10 pts.)

Solve the following initial value problem: $xy' + y^2 = 2xy^2$, $y(1) = 1$.

Problem 3. (10 pts.)

A tank initially contains 100 grams of a radioactive substance. After 1 hour there are 90 grams of the substance remaining in the tank. What is the half-life of the substance? In other words, when will there be 50 grams of the substance remaining in the tank?

Problem 4. (10 pts.) Find the general solution to each of the following linear homogeneous differential equations:

a. (5 pts.) $y^{(4)} - 4y''' + 3y'' = 0$

b. (5 pts.) $y''' - 4y'' + 4y' = 0$

Problem 5. (15 pts.)

Solve the following initial value problem: $y'' + y' - 2y = 8x^2$, $y(0) = 4$, $y'(0) = 0$.

Problem 6. (15 points)

Consider a forced, damped mass-spring system with mass 1 kg, damping coefficient 2 Ns/m, spring constant 9 N/m, and an external force $F_{\text{ext}}(t) = 12 \cos(3t)$ N. Find the steady periodic solution (steady-state solution) for this system.

Problem 7. (10 points)

a. (3 pts.) Find the Laplace transform of $e^{-t} \cos(2t)$

b. (7 pts.) Find the inverse Laplace transform of $\frac{s+1}{s^2-3s+2}$.

Problem 8. (15 points)

Use the Laplace Transform to solve the following IVP: $x'' + x = 2e^t$, $x(0) = 1$, $x'(0) = 0$.

Solutions not using the Laplace transform method will not receive any credit.

Answers to Practice Exam Questions

(Full solutions are available on the course web page under the Course Materials link.)

1. $y = x - 1 + 2e^{-x}$

2. $y = [3 + \ln(x) - 2x]^{-1}$

3. $\frac{\ln(0.5)}{\ln(0.9)} \approx 6.6$ hours

4a. $y = c_1 + c_2x + c_3e^x + c_4e^{3x}$

4b. $y = c_1 + c_2e^{2x} + c_3xe^{2x}$

5. $y = 2e^{-2x} + 8e^x - 4x^2 - 4x - 6$

6. $x_{\text{sp}} = 2 \sin(3t)$

7a. $\frac{s+1}{(s+1)^2+4}$

7b. $-2e^t + 3e^{2t}$

8 $x = e^t - \sin(t)$