92.236 Engineering Differential Equations   Practice Final Exam Solutions
Spring 2015

Problem 1. (15 pts.)

Solve the following initial value problem: $y' + y = x,\ y(0) = 1.$

This is a linear d.e., and it is already in standard form. $\boxed{\text{5 pts.}}$

Find the integrating factor: $\rho(x) = e^{\int 1\ dx} = e^x$. $\boxed{\text{4 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$e^x\,[y' + y] = xe^x$. $\boxed{\text{1 pt.}}$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\,[e^x y] = xe^x$. $\boxed{\text{2 pts.}}$

Integrating both sides, we obtain $e^x y = \displaystyle\int xe^x\ dx = (x-1)e^x + c$ using formula 46 from the Table of Integrals. $\boxed{\text{2 pts.}}$

$y(0) = 1 \Rightarrow e^0(1) = (0-1)e^0 + c \Rightarrow c = 2.$ $\boxed{\text{1 pt.}}$

Therefore, $e^x y = (x-1)e^x + 2$, so $\boxed{\boxed{y = x - 1 + 2e^{-x}.}}$

Problem 2. (10 pts.)

Solve the following initial value problem: $xy' + y^2 = 2xy^2,\ y(1) = 1$

$xy' + y^2 = 2xy^2 \Rightarrow y' = \dfrac{2xy^2 - y^2}{x} = \dfrac{y^2\,(2x-1)}{x}$. This is a separable d.e. $\boxed{\text{3 pts.}}$

Multiply by dx and divide by y^2:

$\dfrac{dy}{y^2} = \dfrac{(2x-1)}{x}\ dx$ $\boxed{\text{2 pts.}}$ $\Rightarrow \displaystyle\int y^{-2}\ dy = \int \left(2 - \frac{1}{x}\right)\ dx \Rightarrow \frac{y^{-1}}{-1} = 2x - \ln(x) + c$ $\boxed{\text{4 pts.}}$

$y(1) = 1 \Rightarrow -1^{-1} = 2(1) - \ln(1) + c \Rightarrow c = -3$ $\boxed{\text{1 pt.}}$

Therefore, $-y^{-1} = 2x - \ln(x) - 3 \Rightarrow y^{-1} = -2x + \ln(x) + 3 \Rightarrow$ $\boxed{\boxed{y = [3 + \ln(x) - 2x]^{-1}}}$

Problem 3. (10 pts.)

A tank initially contains 100 grams of a radioactive substance. After 1 hour there are 90 grams of the substance remaining in the tank. What is the half-life of the substance? In other words, when will there be 50 grams of the substance remaining in the tank?

Let t denote time (in hours) and let x denote the amount (in grams) of radioactive substance in the tank. Then $x = x_0 e^{-kt}$ where $x_0 = x(0)$. $\boxed{\text{5 pts.}}$

$x_0 = 100$ so $x = 100e^{-kt}$ $\boxed{\text{1 pt.}}$

$x(1) = 90 \Rightarrow 90 = 100e^{-k(1)} \Rightarrow 0.9 = e^{-k} \Rightarrow \ln(0.9) = \ln\left(e^{-k}\right) = -k \Rightarrow k = -\ln(0.9)$ $\boxed{\text{2 pts.}}$

Let τ denote the half-life. $x(\tau) = 50 \Rightarrow 50 = 100e^{-k\tau} \Rightarrow 0.5 = e^{-k\tau} \Rightarrow \ln(0.5) = \ln\left(e^{-k\tau}\right) = -k\tau$

$\Rightarrow \tau = -\ln(0.5)/k \Rightarrow$ $\boxed{\boxed{\tau = \dfrac{\ln(0.5)}{\ln(0.9)} \approx 6.6 \text{ hours}}}$ $\boxed{\text{2 pts.}}$

Problem 4. (10 pts.) Find the general solution to each of the following linear homogeneous differential equations:

a. (5 pts.) $y^{(4)} - 4y''' + 3y'' = 0$

The characteristic equation is $r^4 - 4r^3 + 3r^2 = 0 \Rightarrow r^2\left(r^2 - 4r + 3\right) = 0 \Rightarrow r^2(r-1)(r-3) = 0 \Rightarrow$ $r = 0$ (double root) or $r = 1$ or $r = 3$ $\boxed{\text{2 pts.}}$

Therefore, $y = c_1e^{0x} + c_2xe^{0x} + c_3e^{1x} + c_4e^{3x}$, or $\boxed{\boxed{y = c_1 + c_2x + c_3e^x + c_4e^{3x}}}$. $\boxed{\text{3 pts.}}$

b. (5 pts.) $y''' - 4y'' + 4y' = 0$

The characteristic equation is $r^3 - 4r^2 + 4r = 0 \Rightarrow r\left(r^2 - 4r + 4\right) = 0 \Rightarrow r(r-2)^2 = 0 \Rightarrow$ $r = 0$ or $r = 2$ (double root) $\boxed{\text{2 pts.}}$

Therefore, $y = c_1e^{0x} + c_2e^{2x} + c_3xe^{2x}$, or $\boxed{\boxed{y = c_1 + c_2e^{2x} + c_3xe^{2x}}}$. $\boxed{\text{3 pts.}}$

Problem 5. (15 pts.)

Solve the following initial value problem: $y'' + y' - 2y = 8x^2,\ y(0) = 4,\ y'(0) = 0.$

Step 1. Find y_c by solving the d.e. $y'' + y' - 2y = 0$.
Characteristic equation: $r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow r = -2$ or $r = 1$.
Therefore, $y_c = c_1e^{-2x} + c_2e^x$. $\boxed{\text{3 pts.}}$

Step 2. Find y_p. You can use either of the following methods.

Method 1: Undetermined Coefficients. The nonhomogeneous term in the de is $8x^2$, a polynomial of degree 2. We should therefore guess that y_p is a polynomial of degree 2: $y_p = Ax^2 + Bx + C$. No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{\text{3 pts.}}$

$y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A$.
Therefore, the left side of the d.e. is $y'' + y' - 2y = 2A + [2Ax + B] - 2\left[Ax^2 + Bx + C\right] = -2Ax^2 + (2A - 4B)x + (2A + B - 2C)$.
We want this to equal the nonhomogeneous term $8x^2$:
$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = 8x^2 \Rightarrow -2A = 8,\ 2A - 2B = 0,\ 2A + B - 2C = 0$
$\Rightarrow A = -4,\ B = -4,\ C = -6$. Thus, $y_p = -4x^2 - 4x - 6$. $\boxed{\text{6 pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-2x}$ and $y_2 = e^x$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix} = e^{-2x}\left(e^x\right) - \left(-2e^{-2x}\right)\left(e^x\right) = 3e^{-x}. \ \boxed{\text{1 pt.}}$$

$u_1 = \int \frac{-y_2f(x)}{W(x)}\,dx = -\int \frac{e^x\left(8x^2\right)}{3e^{-x}}\,dx = -\frac{8}{3}\int x^2e^{2x}\,dx = -\frac{1}{3}\left(4x^2 - 4x + 2\right)e^{2x}$ using integral table formulas 47 and 46 with $u = 2x$. $\boxed{\text{3 pts.}}$

$u_2 = \int \frac{y_1f(x)}{W(x)}\,dx = \int \frac{e^{-2x}\left(8x^2\right)}{3e^{-x}}\,dx = \frac{8}{3}\int x^2e^{-x}\,dx = -\frac{8}{3}\left(x^2 + 2x + 2\right)e^{-x}$ using integral table formulas 47 and 46 with $u = -x$. $\boxed{\text{3 pts.}}$

Therefore, $y_p = u_1y_1 + u_2y_2 = \left[-\frac{1}{3}\left(4x^2 - 4x + 2\right)e^{2x}\right]e^{-2x} + \left[-\frac{8}{3}\left(x^2 + 2x + 2\right)e^{-x}\right]e^x =$ $-4x^2 - 4x - 6$ $\boxed{\text{1 pt.}}$

Step 3. $y = y_c + y_p$, so $y = c_1e^{-2x} + c_2e^x - 4x^2 - 4x - 6$ $\boxed{\text{1 pt.}}$

Step 4. Use the initial conditions to determine the values of c_1 and c_2.
$y = c_1e^{-2x} + c_2e^x - 4x^2 - 4x - 6 \Rightarrow y' = -2c_1e^{-2x} + c_2e^x - 8x - 4$

$y(0) = 0 \Rightarrow c_1e^0 + c_2e^0 - 4(0)^2 - 4(0) - 6 = 4 \Rightarrow c_1 + c_2 = 10.$
$y'(0) = 0 \Rightarrow -2c_1e^0 + c_2e^0 - 8(0) - 4 = 0 \Rightarrow -2c_1 + c_2 = 4.$

$c_1 + c_2 = 10,\ -2c_1 + c_2 = 4 \Rightarrow c_1 = 2,\ c_2 = 8$ 2 pts. Therefore, $\boxed{y = 2e^{-2x} + 8e^x - 4x^2 - 4x - 6}$

Problem 6. (15 points)

Consider a forced, damped mass-spring system with mass 1 kg, damping coefficient 2 Ns/m, spring constant 9 N/m, and an external force $F_{\text{ext}}(t) = 12\cos(3t)$N. Find the steady periodic solution (steady-state solution) for this system.

The d.e. modeling this system is $mx'' + cx' + kx = F_e(t)$, or $x'' + 2x' + 9x = 12\cos(3t)$. 2 pts.

The steady-state (steady periodic) solution x_{sp} is the particular solution x_p. 3 pts.

You can find x_p using either the Method of Undetermined Coefficients the Method of Variation of Parameters. Here we use the Method of Undetermined Coefficients to save the work of finding x_c.

Since the nonhomogeneous term in the d.e. ($12\cos(3t)$) is a cosine, we guess that x_p is the sum of a cosine and sine with the same frequency: $x_p = A\cos(3t) + B\sin(3t)$. The complementary solution x_c will contain decaying exponential terms because of the damping term in the d.e., so we know that no term in our guess for x_p duplicates a term in x_c. Therefore, there is no need to modify the guess. 4 pts.

$x = A\cos(3t) + B\sin(3t) \Rightarrow x' = -3A\sin(3t) + 3B\cos(3t) \Rightarrow x'' = -9A\cos(3t) - 9B\sin(3t).$

Therefore, the left side of the d.e. is
$x'' + 2x' + 9x = -9A\cos(3t) - 9B\sin(3t) + 2[-3A\sin(3t) + 3B\cos(3t)] + 9[A\cos(3t) + B\sin(3t)] = 6B\cos(3t) - 6A\sin(3t)$. We want this to equal the nonhomogeneous term $12\cos(3t)$, so $6B = 12$ and $-6A = 0 \Rightarrow A = 0$ and $B = 2$. Therefore, $\boxed{x_{\text{sp}} = 2\sin(3t)}$ 6 pts.

Problem 7. (10 points)

a. (3 pts.) Find the Laplace transform of $e^{-t}\cos(2t)$

Using the Laplace transform table entry for $\mathcal{L}\left\{e^{at}\cos(kt)\right\}$ we have $\boxed{\mathcal{L}\left\{e^{-t}\cos(2t)\right\} = \frac{s+1}{(s+1)^2+4}}$

3 pts.

b. (7 pts.) Find the inverse Laplace transform of $\frac{s+1}{s^2-3s+2}$.

Use a partial fraction decomposition: $\frac{s+1}{s^2-3s+2} = \frac{s+1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$ 2 pts.

$(s-1)(s-2)\left[\frac{s+1}{(s-1)(s-2)}\right] = (s-1)(s-2)\left[\frac{A}{s-1} + \frac{B}{s-2}\right] \Rightarrow s+1 = A(s-2) + B(s-1) =$
$(A+B)s + (-2A - B) \Rightarrow A + B = 1,\ -2A - B = 1 \Rightarrow A = -2,\ B = 3$ 3 pts.

Therefore, $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{-2}{s-1} + \frac{3}{s-2}\right\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = \boxed{-2e^t + 3e^2t}$

2 pts.

Problem 8. (15 points)

Use the Laplace Transform to solve the following IVP: $x'' + x = 2e^t,\ x(0) = 1,\ x'(0) = 0.$

Solutions not using the Laplace transform method will not receive any credit.

$$x'' + x = 2e^t \Rightarrow \mathcal{L}\left\{x'' + x\right\} = \mathcal{L}\left\{2e^t\right\} \Rightarrow \mathcal{L}\left\{x''\right\} - \mathcal{L}\left\{x'\right\} = 2\mathcal{L}\left\{e^t\right\} = \frac{2}{s-1} \boxed{\text{3 pts.}}$$

$$\Rightarrow \left[s^2\mathcal{L}\{x\} - sx(0) - x'(0)\right] + \mathcal{L}\{x\} = \frac{2}{s-1} \boxed{\text{3 pts.}}$$

$$\left[s^2\mathcal{L}\{x\} - s \cdot 1 - 0\right] + \mathcal{L}\{x\} = \frac{2}{s-1} \Rightarrow \left(s^2 + 1\right)\mathcal{L}\{x\} = \frac{2}{s-1} + s = \frac{2 + s^2 - s}{s-1} = \frac{s^2 - s + 2}{s-1} \Rightarrow$$

$$\mathcal{L}\{x\} = \frac{s^2 - s + 2}{(s-1)\left(s^2+1\right)} \boxed{\text{1 pt.}} \Rightarrow x = \mathcal{L}^{-1}\left\{\frac{s^2 - s + 2}{(s-1)\left(s^2+1\right)}\right\}.$$

Use a partial fraction decomposition: $\dfrac{s^2 - s + 2}{(s-1)\left(s^2+1\right)} = \dfrac{A}{s-1} + \dfrac{Bs + C}{s^2 + 1}$

$$(s-1)\left(s^2+1\right)\left[\frac{s^2 - s + 2}{(s-1)\left(s^2+1\right)}\right] = (s-1)\left(s^2+1\right)\left[\frac{A}{s-1} + \frac{Bs + C}{s^2+1}\right] \Rightarrow$$

$$s^2 - s + 2 = A\left(s^2 + 1\right) + (Bs + C)(s-1) = As^2 + A + Bs^2 + Cs - Bs - C = (A+B)s^2 + (C-B)s + (A-C)$$

$$\Rightarrow A + B = 1,\ C - B = -1,\ A - C = 2 \Rightarrow A = 1,\ B = 0,\ C = -1. \boxed{\text{6 pts.}}$$

Therefore, $x = \mathcal{L}^{-1}\left\{\dfrac{1}{s-1} + \dfrac{0s - 1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} \Rightarrow \boxed{\boxed{x = e^t - \sin(t)}} \boxed{\text{2 pts.}}$