92.236 Engineering Differential Equations Practice Exam # 1 Solutions
Spring 2015

Problem 1. (10 points)

Is $y = x^4$ a solution of the d.e. $xy' = y^2 + 4x^4$? Why or why not?

Left side of d.e: $y = x^4 \Rightarrow y' = 4x^3 \Rightarrow xy' = x\left(4x^3\right) = 4x^4$. $\boxed{\text{3 pts.}}$

Right side of d.e: $y = x^4 \Rightarrow y^2 + 4x^4 = \left(x^4\right)^2 + 4x^4 = x^8 + 4x^4$. $\boxed{\text{3 pts.}}$

Left side $\neq$ right side, so $\boxed{\boxed{y = x^4 \textbf{ is not} \text{ a solution of the d.e. } xy' = y^2 + 4x^4}}$ $\boxed{\text{4 pts.}}$

Problem 2. (15 points)

A car is traveling at 25 m/s when its brakes are applied. The car stops after traveling 100 meters. How long (in seconds) did it take the car to stop after the brakes were applied? Assume the car's deceleration was constant.

Let t denote time in seconds, with $t = 0$ corresponding to the time the brakes are applied. Let x denote the position of the car (in meters) at time t, with $x = 0$ corresponding to the position at which the brakes were applied. Let v denote the velocity of the car (in meters per second) at time t. Then $\boxed{\text{5 pts.}}$ $v = at + v_0 = at + 25$ and $x = \frac{1}{2}at^2 + v_0 t + x_0 = \frac{1}{2}at^2 + 25t$. Let T denote the time at which the car comes to a stop. Then $v(T) = 0$ and $x(T) = 100$, so $0 = aT + 25$ and $100 = \frac{1}{2}aT^2 + 25T$ $\boxed{\text{5 pts.}}$ $\Rightarrow a = -\frac{25}{T}$ and $100 = \frac{1}{2}\left(-\frac{25}{T}\right)T^2 + 25T = \frac{25T}{2} \Rightarrow$ $25T = 200 \Rightarrow T = 8$ $\boxed{\text{5 pts.}}$. Therefore, $\boxed{\text{it takes 8 seconds for the car to stop.}}$

Problem 3. (25 points)

Solve the following initial value problem.

$$\frac{dy}{dx} = \frac{\cos(x)}{3y^2}, \quad y(0) = 3.$$

This is a separable d.e. $\boxed{\text{5 pts.}}$

$\frac{dy}{dx} = \frac{\cos(x)}{3y^2} \Rightarrow dy = \frac{\cos(x)}{3y^2}\,dx \Rightarrow 3y^2\,dy = \cos(x)\,dx.$ $\boxed{\text{5 pts.}}$

$\Rightarrow \int 3y^2\,dy = \int \cos(x)\,dx \Rightarrow y^3 = \sin(x) + c.$ $\boxed{\text{12 pts.}}$

$y(0) = 3 \Rightarrow 3^3 = \sin(0) + c \Rightarrow c = 27$ $\boxed{\text{3 pts.}}$ $\Rightarrow y^3 = \sin(x) + 27 \Rightarrow$ $\boxed{\boxed{y = [\sin(x) + 27]^{1/3}}}$.

Problem 4. (25 points)

Solve the following initial value problem.

$$\frac{dy}{dx} = 8 - \frac{3y}{x}, \quad y(1) = 3.$$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone. $\boxed{\text{5 pts.}}$

First write the equation in standard form:

$\dfrac{dy}{dx} = 8 - \dfrac{3y}{x} \Rightarrow \dfrac{dy}{dx} + \left(\dfrac{3}{x}\right) y = 8$ $\boxed{\text{3 pts.}}$

Next, find the integrating factor: $\rho(x) = e^{\int 3/x \ dx} = e^{3\ln(x)} = x^3$. $\boxed{\text{6 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^3 \left[\dfrac{dy}{dx} + \left(\dfrac{3}{x}\right) y\right] = x^3 \cdot 8 \Rightarrow x^3 \dfrac{dy}{dx} + 3x^2 y = 8x^3$. $\boxed{\text{2 pts.}}$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\left[x^3 y\right] = 8x^3$. $\boxed{\text{4 pts.}}$

Integrating both sides, we obtain $x^3 y = \displaystyle\int 8x^3 \ dx = 2x^4 + c$. $\boxed{\text{3 pts.}}$

$y(1) = 3 \Rightarrow 1^3(3) = 2(1)^2 + c \Rightarrow c = 1$ $\boxed{\text{2 pts.}}$

Therefore, $x^3 y = 2x^4 + 1$, so $\boxed{\boxed{y = 2x + x^{-3}.}}$

Problem 5. (15 points)

A tank initially contains 100 liters of water in which 80 grams of salt are dissolved. A salt solution containing 10 grams of salt per liter is pumped into the tank at the rate of 4 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 6 liters per minute.

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in grams). Write down the differential equation $\left(\dfrac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.

$\dfrac{dx}{dt} = \text{rate in - rate out}$ $\boxed{\text{3 pts.}}$

$=$ (flow rate in)(concentration in) - (flow rate out)(concentration out), $\boxed{\text{3 pts.}}$ so

$$\frac{dx}{dt} = \underbrace{\left(4\frac{\text{liters}}{\text{minute}}\right)}_{\boxed{\text{1 pt.}}} \underbrace{\left(10\frac{\text{gm}}{\text{liter}}\right)}_{\boxed{\text{1 pt.}}} - \underbrace{\left(6\frac{\text{liters}}{\text{minute}}\right)}_{\boxed{\text{1 pt.}}} \underbrace{\left(\frac{x \text{ gm}}{(100-2t) \text{ liters}}\right)}_{\boxed{\text{5 pts.}}}.$$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 100 + (4-6)t$ liters.)

Initially there 80 gm of salt in the tank, so $x(0) = 80$ $\boxed{\text{1 pt.}}$.

Therefore, the initial value problem describing this mixing problem is $\boxed{\boxed{\dfrac{dx}{dt} = 40 - \dfrac{6x}{100-2t} \quad \text{with } x(0) = 80.}}$