**92.236 Engineering Differential Equations  Practice Exam # 2 Solutions**
**Spring 2015**

**Problem 1. (20 points)** Consider the autonomous differential equation $\dfrac{dx}{dt} = x^3 + 3x^2$.

a. Find all critical points (equilibrium solutions) of this d.e.

$x^3 + 3x^2 = 0 \Rightarrow x^2(x+3) = 0 \Rightarrow$ $\boxed{\boxed{\text{the critical points are } -3 \text{ and } 0}}$ $\boxed{3 \text{ pts.}}$

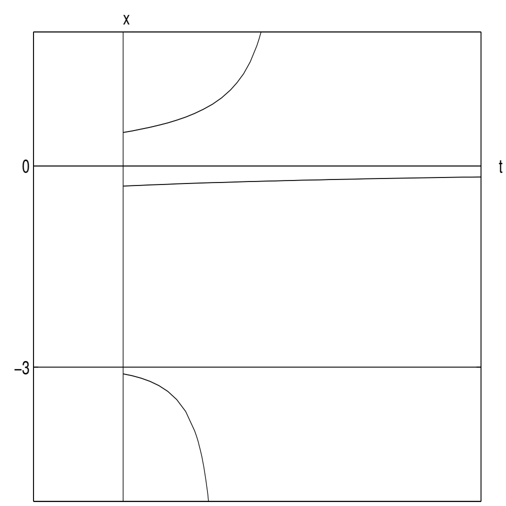
b. Draw the phase line (phase diagram) for this d.e. $\boxed{8 \text{ pts.}}$

The three critical points divide the phase line into 3 intervals: $x > 0$, $-3 < x < 0$, and $x < -3$.

$\left.\dfrac{dx}{dt}\right|_{x=1} = 1^2(1+3) > 0$, so the direction arrow points up for $x > 0$.

$\left.\dfrac{dx}{dt}\right|_{x=-1} = (-1)^2(-1+3) > 0$, so the direction arrow points up for $-3 < x < 0$.

$\left.\dfrac{dx}{dt}\right|_{x=-4} = (-4)^2(-4+3) < 0$, so the direction arrow points down for $x < -3$.


c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\boxed{\text{both 0 and } -3 \text{ are unstable}}}$. $\boxed{2 \text{ pts.}}$

d. If $x(0) = -2$, what value will $x(t)$ approach as $t$ increases?

Since $-2$ lies in the interval $-3 < x < 0$, we can see from the phase line that $\boxed{\boxed{x(t) \to 0}}$ as $t$ increases. $\boxed{3 \text{ pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. $\boxed{4 \text{ pts.}}$

See the figure above.

**Problem 2. (20 points)** Solve the following differential equations:

a. $y'' + 8y' + 15y = 0$

Characteristic equation: $r^2 + 8r + 15 = 0 \Rightarrow (r+5)(r+3) = 0 \Rightarrow$
$r = -5$ or $r = -3$. $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-5x} + c_2 e^{-3x}}}$ $\boxed{\text{6 pts.}}$

b. $y'' + 6y' + 9y = 0$

Characteristic equation: $r^2 + 6r + 9 = 0 \Rightarrow (r+3)^2 = 0 \Rightarrow$
$r = -3$ (repeated root). $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-3x} + c_2 x e^{-3x}}}$ $\boxed{\text{6 pts.}}$

**Problem 3. (20 points)** Solve the following initial value problem.

$$x^4 - 4y^4 + 4xy^3 \frac{dy}{dx} = 0, \quad y(1) = 2.$$

$x^4 - 4y^4 + 4xy^3 \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{4y^4 - x^4}{4xy^3}$. $dy/dx$ equals a rational function, and every term has the same degree (4). Therefore, this d.e. is homogeneous. $\boxed{\text{4 pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace $y$ by $xv$:

$$\frac{dy}{dx} = \frac{4y^4 - x^4}{4xy^3} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{4(xv)^4 - x^4}{4x(xv)^3}}_{\boxed{\text{4 pts.}}} = \frac{x^4(4v^4 - 1)}{4x^4v^3} = \frac{4v^4 - 1}{4v^3} = v - \frac{1}{4v^3} \Rightarrow \underbrace{x\frac{dv}{dx} = \frac{-1}{4v^3}}_{\boxed{\text{3 pts.}}}$$

$$\underbrace{\Rightarrow 4v^3\, dv = \frac{-1}{x}\, dx}_{\boxed{\text{2 pts.}}} \Rightarrow \int 4v^3\, dv = \int \frac{-1}{x}\, dx \Rightarrow \underbrace{v^4 = -\ln(x) + c}_{\boxed{\text{3 pts.}}} \Rightarrow \underbrace{\left(\frac{y}{x}\right)^4 = -\ln(x) + c}_{\boxed{\text{2 pts.}}}$$

The initial condition $y(1) = 2 \Rightarrow (2/1)^4 = -\ln(1) + c \Rightarrow c = 16$ $\boxed{\text{2 pts.}}$.

Therefore, $\left(\dfrac{y}{x}\right)^4 = -\ln(x) + 16 \Rightarrow \dfrac{y}{x} = [16 - \ln(x)]^{1/4} \Rightarrow \boxed{\boxed{y = x\,[16 - \ln(x)]^{1/4}}}$.

**Problem 4. (20 points)** Solve the following initial value problem.

$$2xy + 4x^3 + \left[x^2 + 6y^2\right] \frac{dy}{dx} = 0, \quad y(-1) = 1.$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{2xy+4x^3}_{M}+\underbrace{\left[x^2+6y^2\right]}_{N}\frac{dy}{dx}=0$$

$\dfrac{\partial M}{\partial y}=\dfrac{\partial}{\partial y}\left[2xy+4x^3\right]=2x.$ $\boxed{\text{1 pt.}}$ $\dfrac{\partial N}{\partial x}=\dfrac{\partial}{\partial x}\left[x^2+6y^2\right]=2x.$ $\boxed{\text{1 pt.}}$

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{3 pts.}}$ Therefore, the solution of the d.e. is $f(x,y)=c$, where the function $f$ satisfies the conditions $\dfrac{\partial f}{\partial x}=M=2xy+4x^3$ and $\dfrac{\partial f}{\partial y}=N=x^2+6y^2$.

$\dfrac{\partial f}{\partial x}=2xy+4x^3\Rightarrow f=\displaystyle\int\left(2xy+4x^3\right)\,\partial x=x^2y+x^4+g(y)$ $\boxed{\text{6 pts.}}$

$\Rightarrow\dfrac{\partial f}{\partial y}=\dfrac{\partial}{\partial y}\left[x^2y+x^4+g(y)\right]=x^2+g'(y)$

But $\dfrac{\partial f}{\partial y}=N=x^2+6y^2\Rightarrow x^2+g'(y)=x^2+6y^2\Rightarrow g'(y)=6y^2\Rightarrow g(y)=2y^3\Rightarrow$

$f=x^2y+x^4+2y^3$ $\boxed{\text{6 pts.}}$

Therefore, the solution of the d.e. is $x^2y+x^4+2y^3=c$ $\boxed{\text{2 pts.}}$

$y(-1)=1\Rightarrow(-1)^2(1)+(-1)^4+2(1)^3=c\Rightarrow c=4.$ $\boxed{\text{1 pt.}}$

Therefore, the solution of the initial value problem is $\boxed{\boxed{x^2y+x^4+2y^3=4}}$

**Problem 5. (10 points)** Let $P$ denote the population of a colony of tribbles. Suppose that $\beta$ (the number of births per week per tribble) is proportional to $P$ and that $\delta$ (the number of deaths per week per tribble) equals 0. Suppose the initial population is 2 and the population after 5 weeks is 4. When will the population reach 20?

$\dfrac{dP}{dt}=\beta P-\delta P$ $\boxed{\text{1 pt.}}$

From the given information we have $\beta=kP$ and $\delta=0$ where $k$ is a positive constant. $\boxed{\text{1 pt.}}$

Therefore, $\dfrac{dP}{dt}=(kP)\,P-(0)(P)\Rightarrow\dfrac{dP}{dt}=kP^2$ $\boxed{\text{1 pt.}}$

This is a separable d.e: $\dfrac{dP}{dt}=kP^2\Rightarrow\dfrac{dP}{P^2}=k\ dt\Rightarrow\displaystyle\int P^{-2}\ dP=\int k\ dt\Rightarrow\dfrac{P^{-1}}{-1}=kt+c\Rightarrow$

$-P^{-1}=kt+c.$ $\boxed{\text{4 pts.}}$

$P(0)=2\Rightarrow-2^{-1}=k(0)+c\Rightarrow c=-1/2\Rightarrow-P^{-1}=kt-1/2$ $\boxed{\text{1 pt.}}$

$P(5)=4\Rightarrow-4^{-1}=k(5)-1/2\Rightarrow 5k=1/2-1/4=1/4\Rightarrow k=1/20\Rightarrow-P^{-1}=t/20-1/2$

$\boxed{\text{1 pt.}}$

$P(t)=20\Rightarrow-20^{-1}=t/20-1/2\Rightarrow-1/20=t/20-1/2\Rightarrow t/20=1/2-1/20=9/20\Rightarrow$ $\boxed{\boxed{t=9\text{ weeks}}}$

$\boxed{\text{1 pt.}}$