92.236 Engineering Differential Equations Practice Exam # 3 Solutions
Spring 2015

Problem 1. (20 pts.) Solve the following differential equations.

a. (8 pts.) $y'' + 4y' + 8y = 0$

Characteristic equation: $r^2 + 4r + 8 = 0 \Rightarrow r = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2(1)} = \dfrac{-4 \pm \sqrt{-16}}{2} = \dfrac{-4 \pm 4i}{2} = -2 \pm 2i$

$\boxed{\text{3 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-2x}\cos(2x) + c_2 e^{-2x}\sin(2x)}}$ $\boxed{\text{5 pts.}}$

b. (12 pts.) $y^{(4)} + 4y'' = 0$

Characteristic equation: $r^4 + 4r^2 = 0 \Rightarrow r^2\left(r^2 + 4\right) = 0 \Rightarrow r = 0$ (double root), $r = \pm 2i = 0 \pm 2i$ $\boxed{\text{4 pts.}}$

Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{0x}\cos(2x) + c_4 e^{0x}\sin(2x)$ or $\boxed{\boxed{y = c_1 + c_2 x + c_3\cos(2x) + c_4\sin(2x)}}$

$\boxed{\text{8 pts.}}$

Problem 2. (25 pts.) Solve the following initial value problem:

$$y'' + y' = 3 + 4\sin(x), \quad y(0) = 0, \quad y'(0) = 1.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' + y' = 0$. Characteristic equation: $r^2 + r = 0 \Rightarrow r(r+1) = 0 \Rightarrow r = 0$ or $r = -1$. Therefore, $y_c = c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x}$. $\boxed{\text{5 pts.}}$

Step 2. Find y_p. You can use either of the following methods.

Method 1: Undetermined Coefficients. The nonhomogeneous term in the de is $3 + 4\sin(x)$. To handle the constant 3, a polynomial of degree 0, we should guess that y_p contains a polynomial of degree 0: A. This guess duplicates the term c_1 in y_c, so we should multiply this part of the guess by x to get Ax.

To handle the $4\sin(x)$ term, we should guess that y_p contains a combination of sine and cosine with the same coefficient of x: $B\sin(x) + C\cos(x)$. This guess does not duplicate any term in y_c, so there is no need to multiply this part of the guess by x.

Therefore, the complete guess is $y_p = Ax + B\sin(x) + C\cos(x)$. $\boxed{\text{7 pts.}}$

$y = Ax + B\sin(x) + C\cos(x) \Rightarrow y' = A + B\cos(x) - C\sin(x) \Rightarrow y'' = -B\sin(x) - C\cos(x)$.
Therefore, the left side of the d.e. is $y'' + y' = [-B\sin(x) - C\cos(x)] + [A + B\cos(x) - C\sin(x)] = A + (-B - C)\sin(x) + (B - C)\cos(x)$
We want this to equal the nonhomogeneous term $3 + 4\sin(x)$:
$A + (-B - C)\sin(x) + (B - C)\cos(x) = 3 + 4\sin(x) \Rightarrow A = 3,\ -B - C = 4,\ B - C = 0$
$\Rightarrow A = 3,\ B = -2,\ C = -2$. Thus, $y_p = 3x - 2\sin(x) - 2\cos(x)$. $\boxed{\text{8 pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{0x} = 1$ and $y_2 = e^{-x}$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = 1\left(-e^{-x}\right) - 0\left(e^{-x}\right) = -e^{-x}$. $\boxed{\text{2 pts.}}$

$u_1 = \displaystyle\int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{-x}\left(3 + 4\sin(x)\right)}{-e^{-x}}\,dx = \int \left(3 + 4\sin(x)\right)\,dx = 3x - 4\cos(x)$. $\boxed{\text{5 pts.}}$

$u_2 = \displaystyle\int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{1\left(3 + 4\sin(x)\right)}{-e^{-x}}\,dx = -\int e^x\left(3 + 4\sin(x)\right)\,dx = -\left\{-3e^x + 2e^x\left[\sin(x) - \cos(x)\right]\right\} =$
$e^x\left[3 - 2\sin(x) + 2\cos(x)\right]$ using formula 49 from the integral table). $\boxed{\text{5 pts.}}$

Therefore, $y_p = u_1y_1 + u_2y_2 = [3x - 4\cos(x)](1) + \{e^x[3 - 2\sin(x) + 2\cos(x)]\}(e^{-x}) =$
$3x - 4\cos(x) + 3 - 2\sin(x) + 2\cos(x) = 3x - 2\sin(x) - 2\cos(x) + 3$ $\boxed{\text{2 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1 + c_2e^{-x} + 3x - 2\sin(x) - 2\cos(x)$ $\boxed{\text{2 pts.}}$ (If you used the Method of Variation of Parameters, you can combine the 3 from y_p with the c_1 from y_c.)

Step 4. Use the initial conditions to determine the values of c_1 and c_2.
$y = c_1 + c_2e^{-x} + 3x - 2\sin(x) - 2\cos(x) \Rightarrow y' = -c_2e^{-x} + 3 - 2\cos(x) + 2\sin(x)$
$y(0) = 0 \Rightarrow c_1 + c_2e^0 + 3(0) - 2\sin(0) - 2\cos(0) = 0 \Rightarrow c_1 + c_2 = 2.$
$y'(0) = 1 \Rightarrow -c_2e^0 + 3 - 2\cos(0) + 2\sin(0) = 1 \Rightarrow -c_2 + 1 = 1 \Rightarrow c_2 = 0.$
$c_1 + c_2 = 2,\ c_2 = 0 \Rightarrow c_1 = 2,\ c_2 = 0.$ $\boxed{\text{3 pts.}}$ Therefore, $\boxed{\boxed{y = 2 + 3x - 2\sin(x) - 2\cos(x)}}$

Problem 3. (20 pts.) Consider a free (unforced), damped mass-spring system with mass $m = 1$ kg, damping constant $c = 4$ N·s/m, and spring constant $k = 5$ N/m. Assume that $x(0) = -1$ m and $x'(0) = 3$ m/s.

a. (16 pts.) Find the position function $x(t)$.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_e(t)$. $\boxed{\text{3 pts.}}$

In this problem, $m = 1$, $c = 4$, $k = 5$, and $F_e(t) = 0$. Therefore, we have $x'' + 4x' + 5x = 0$. $\boxed{\text{2 pts.}}$

The characteristic equation for this d.e. is $r^2 + 4r + 5 = 0 \Rightarrow$
$r = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} = \dfrac{-4 \pm \sqrt{-4}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i.$ $\boxed{\text{4 pts.}}$

Therefore, $x = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t)$. $\boxed{\text{5 pts.}}$

$x = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t) \Rightarrow x' = c_1\left[-2e^{-2t}\cos(t) - e^{-2t}\sin(t)\right] + c_2\left[-2e^{-2t}\sin(t) + e^{-2t}\cos(t)\right].$
$x(0) = -1$ and $x'(0) = 3 \Rightarrow -1 = c_1e^0\cos(0) + c_2e^0\sin(0) = c_1$ and $3 = c_1\left[-2e^0\cos(0) - e^0\sin(0)\right] + c_2\left[-2e^0\sin(0) + e^0\cos(0)\right] = -2c_1 + c_2 \Rightarrow c_1 = -1, c_2 = 1$. Therefore, $\boxed{\boxed{x = -e^{-2t}\cos(t) + e^{-2t}\sin(t)}}$
$\boxed{\text{2 pts.}}$

b. (4 pts.) Write the solution from part a in the form $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$

$x = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t) = e^{-2t}[c_1\cos(t) + c_2\sin(t)] = e^{-2t}[C\cos(t - \alpha)]$
$C = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ $\boxed{\text{2 pts.}}$
Because $c_1 < 0$ we have $\alpha = \pi + \tan^{-1}(c_2/c_1) = \pi + \tan^{-1}(-1) = \pi - \pi/4 = \frac{3\pi}{4}$ $\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{x = \sqrt{2}e^{-2t}\cos\left(t - \dfrac{3\pi}{4}\right)}}$

Problem 4. (20 pts.) Solve the system $\begin{cases} x' &= y \\ y' &= -x + 2y \end{cases}$

Take the derivative of both sides of the first d.e. in the system: $x' = y \Rightarrow x'' = y'$. The second d.e. in the system is $y' = -x + 2y$. Therefore, $x'' = -x + 2y$. From the first d.e. in the system, $y = x'$, so we have $x'' = -x + 2x'$ $\boxed{\text{8 pts.}}$

$x'' = -x + 2x' \Rightarrow x'' - 2x' + x = 0.$
Characteristic equation: $r^2 - 2r + 1 = 0 \Rightarrow (r - 1)^2 = 0 \Rightarrow r = 1$ (double root) $\Rightarrow x = c_1e^t + c_2te^t$. $\boxed{\text{8 pts.}}$
The first d.e. in the given system says $y = x'$, so $y = c_1e^t + c_2\left[e^t + te^t\right]$. Therefore, the solution of the given system is $\boxed{\boxed{x = c_1e^t + c_2te^t,\ y = c_1e^t + c_2\left[e^t + te^t\right]}}$ $\boxed{\text{4 pts.}}$