Problem 1. (10 points)

Is $y(x) = x^3$ is a solution of the d.e. $x^4 y' = x^6 + 2y^2$? Why or why not?

Left side of d.e: $y(x) = x^3 \Rightarrow y' = 3x^2 \Rightarrow x^4 y' = x^4\left(3x^2\right) = 3x^6$. $\boxed{\text{3 pts.}}$

Right side of d.e: $y = x^3 \Rightarrow x^6 + 2y^2 = x^6 + 2\left(x^3\right)^2 = x^6 + 2x^6 = 3x^6$. $\boxed{\text{3 pts.}}$

Left side = right side, so $\boxed{\boxed{y(x) = x^2 \textbf{ is} \text{ a solution of the d.e. } x^4 y' = x^6 + 2y^2}}$ $\boxed{\text{4 pts.}}$

Problem 2. (15 points)

A container initially holds 50 grams of a radioactive substance. There are 45 grams of the substance left after 2 hours. When will there be 20 grams of the substance left in the container?

Let t denote time in hours, and let x denote the amount (in grams) of radioactive substance in the container at time t. Then $x = x_0 e^{-kt}$, where $x_0 = x(0)$.

We are told that $x_0 = 50$ so $x = 50e^{-kt}$ $\boxed{\text{5 pts.}}$

$x(2) = 45 \Rightarrow 45 = 50e^{-k(2)} = e^{-2k} \Rightarrow 45/50 = e^{-2k} \Rightarrow \ln(0.9) = \ln\left(e^{-2k}\right) = -2k \Rightarrow k = -\ln(0.9)/2$

$\boxed{\text{5 pts.}}$

To find the time when $x = 20$, set $x = 20$ in the equation $x = 50e^{-kt}$ and solve for t:

$$20 = 50e^{-kt} \Rightarrow 20/50 = e^{-kt} \Rightarrow \ln(0.4) = \ln\left(e^{-kt}\right) = -kt \Rightarrow t = \frac{\ln(0.4)}{-k} \Rightarrow \boxed{\boxed{t = 2\frac{\ln(0.4)}{\ln(0.9)} \approx 17.4 \text{ hours}}}.$$

$\boxed{\text{5 pts.}}$

Problem 3. (25 points)

Solve the following initial value problem.

$$\frac{dy}{dx} = \frac{x^2 + 2y}{x}, \quad y(1) = 2.$$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone. $\boxed{\text{5 pts.}}$

First write the equation in standard form:

$\dfrac{dy}{dx} = \dfrac{x^2 + 2y}{x} = x + \dfrac{2y}{x} \Rightarrow \dfrac{dy}{dx} - \left(\dfrac{2}{x}\right) y = x$ $\boxed{\text{3 pts.}}$

Next, find the integrating factor: $\rho(x) = e^{\int -2/x \, dx} = e^{-2\ln(x)} = x^{-2}$. $\boxed{\text{6 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^{-2}\left[\dfrac{dy}{dx} - \left(\dfrac{2}{x}\right) y\right] = x^{-2}(x) \Rightarrow x^{-2}\dfrac{dy}{dx} - 2x^{-3} y = x^{-1}$. $\boxed{\text{2 pts.}}$

Use the Product Rule backwards to rewrite the d.e. as $\frac{d}{dx}\left[x^{-2}y\right] = x^{-1}$. $\boxed{\text{3 pts.}}$

Integrating both sides, we obtain $x^{-2}y = \int x^{-1}\ dx = \ln(x) + c$. $\boxed{\text{3 pts.}}$

$y(1) = 2 \Rightarrow 1^{-2}(2) = \ln(1) + c \Rightarrow c = 2$ $\boxed{\text{3 pts.}}$

Therefore, $x^{-2}y = \ln(x) + 2$, so $\boxed{\boxed{y = x^2\ln(x) + 2x^2.}}$

Problem 4. (25 points)

Solve the following initial value problem.

$$\frac{dy}{dx} = \frac{2x}{1+2y}, \quad y(2) = 1.$$

This is a separable d.e. $\boxed{\text{5 pts.}}$

$\frac{dy}{dx} = \frac{2x}{1+2y} \Rightarrow (1+2y)\ dy = 2x\ dx.$ $\boxed{\text{5 pts.}}$

$\Rightarrow \int (1+2y)\ dy = \int 2x\ dx \Rightarrow y + y^2 = x^2 + c.$ $\boxed{\text{12 pts.}}$

$y(2) = 1 \Rightarrow 1 + 1^2 = 2^2 + c \Rightarrow c = -2$ $\boxed{\text{3 pts.}}$ $\Rightarrow y + y^2 = x^2 - 2 \Rightarrow \boxed{\boxed{y = \frac{-1+\sqrt{4x^2-7}}{2}}}$.

Problem 5. (15 points)

A tank initially contains 50 liters of water in which 20 grams of salt are dissolved. A salt solution containing 10 grams of salt per liter is pumped into the tank at the rate of 6 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 3 liters per minute.

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in grams). Write down the differential equation $\left(\frac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.

$\frac{dx}{dt} = \text{rate in - rate out}$ $\boxed{\text{3 pts.}}$

$=$ (flow rate in)(concentration in) - (flow rate out)(concentration out), $\boxed{\text{3 pts.}}$ so

$$\frac{dx}{dt} = \underbrace{\left(6\frac{\text{liters}}{\text{minute}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(10\frac{\text{gm}}{\text{liter}}\right)}_{\boxed{\text{1 pt.}}} - \underbrace{\left(3\frac{\text{liters}}{\text{minute}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(\frac{x\text{ gm}}{(50+3t)\text{ liters}}\right)}_{\boxed{\text{5 pts.}}}.$$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 50 + (6-3)t$ liters.)

Initially there are 20 gm. of salt in the tank, so $x(0) = 20$ $\boxed{\text{1 pt.}}$.

Therefore, the initial value problem describing this mixing problem is $\boxed{\boxed{\frac{dx}{dt} = 60 - \frac{3x}{50+3t} \quad \text{with } x(0) = 20.}}$