92.236 Engineering Differential Equations Exam # 2 Solutions Spring 2011

Problem 1. (20 points) Consider the autonomous differential equation $\dfrac{dx}{dt} = 4x^2 - x^4$.

a. Find all critical points (equilibrium solutions) of this d.e.

$4x^2 - x^4 = 0 \Rightarrow x^2\left(4 - x^2\right) = 0 \Rightarrow x^2(2 + x)(2 - x) = 0 \Rightarrow$
$\boxed{\boxed{\text{the critical points are } -2, 0 \text{ and } 2}}$ $\boxed{\text{3 pts.}}$

b. Draw the phase line (phase diagram) for this d.e. $\boxed{\text{8 pts.}}$

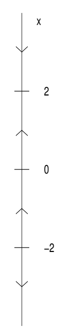
The three critical points divide the phase line into 4 intervals: $x > 2$, $0 < x < 2$, $-2 < x < 0$, and $x < -2$.

$\left.\dfrac{dx}{dt}\right|_{x=3} = 3^2(2 + 3)(2 - 3) < 0$, so the direction arrow points down for $x > 2$.

$\left.\dfrac{dx}{dt}\right|_{x=1} = 1^2(2 + 1)(2 - 1) > 0$, so the direction arrow points up for $0 < x < 2$.

$\left.\dfrac{dx}{dt}\right|_{x=-1} = (-1)^2(2 - 1)(2 - (-1)) > 0$, so the arrow points up for $-2 < x < 0$.

$\left.\dfrac{dx}{dt}\right|_{x=-3} = (-3)^2(2 - 3)(2 - (-3)) < 0$, so the arrow points down for $x < -2$.



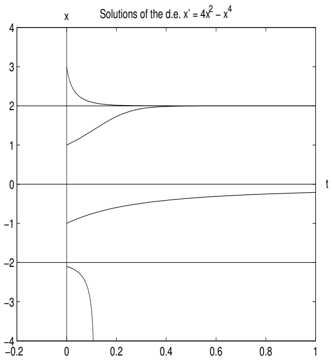
c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\boxed{\text{2 is stable but } -2 \text{ and } 0 \text{ are unstable}}}$. $\boxed{\text{3 pts.}}$

d. If $x(0) = -1$, what value will $x(t)$ approach as t increases?

Since -1 lies in the interval $-2 < x < 0$, we can see from the phase line that $\boxed{\boxed{x(t) \to 0}}$ as t increases. $\boxed{\text{3 pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. $\boxed{\text{3 pts.}}$



Problem 2. (20 points) Solve the following differential equations:

a. $y'' - 10y' + 25y = 0$

Characteristic equation: $r^2 - 10r + 25 = 0 \Rightarrow (r-5)^2 = 0 \Rightarrow$
$r = 5$ (repeated root). $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}}$ $\boxed{\text{6 pts.}}$

b. $y'' + y' - 2y = 0$

Characteristic equation: $r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow$
$r = -2$ or $r = 1$. $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-2x} + c_2 e^{x}}}$ $\boxed{\text{6 pts.}}$

Problem 3. (20 points) Solve the following initial value problem.

$$3x^2 + 6xy + \left[3x^2 + 2y\right] \frac{dy}{dx} = 0, \quad y(1) = 0.$$

$$\underbrace{3x^2 + 6xy}_{M} + \underbrace{\left[3x^2 + 2y\right]}_{N} \frac{dy}{dx} = 0.$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[3x^2 + 6xy\right] = 6x.$ $\boxed{\text{1 pt.}}$ $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[3x^2 + 2y\right] = 6x.$ $\boxed{\text{1 pt.}}$
Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{3 pts.}}$ Therefore, the solution of the d.e. is $f(x,y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 3x^2 + 6xy$ and $\dfrac{\partial f}{\partial y} = N = 3x^2 + 2y$.
$\dfrac{\partial f}{\partial x} = 3x^2 + 6xy \Rightarrow f = \displaystyle\int \left(3x^2 + 6xy\right)\, dx = x^3 + 3x^2 y + g(y)$ $\boxed{\text{6 pts.}}$
$\Rightarrow \dfrac{\partial f}{\partial y} = 3x^2 + g'(y)$. But $\dfrac{\partial f}{\partial y} = N = 3x^2 + 2y$, so $3x^2 + g'(y) = 3x^2 + 2y \Rightarrow g'(y) = 2y \Rightarrow$
$g(y) = y^2$ $\boxed{\text{5 pts.}}$. Therefore, $f = x^3 + 3x^2 y + y^2$, so the solution of the d.e. is

$x^3+3x^2y+y^2=c$. $\boxed{\text{3 pts.}}$ The initial condition $y(1)=0 \Rightarrow 1^3+3(1)^2(0)+0^2=c \Rightarrow c=1$ $\boxed{\text{1 pt.}}$.

Therefore, the solution of the given IVP is $\boxed{\boxed{x^3+3x^2y+y^2=1}}$

Problem 4. (20 points) Solve the following initial value problem.

$$x^3+y^3-xy^2\frac{dy}{dx}=0, \quad y(1)=3.$$

$x^3+y^3-xy^2\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=\dfrac{x^3+y^3}{xy^2}$. Since dy/dx equals a rational function in which each term has the same degree (3), the d.e. is homogeneous $\boxed{\text{5 pts.}}$. We introduce the new variable $v=y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v+x\dfrac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx}=\frac{x^3+y^3}{xy^2} \Rightarrow \underbrace{v+x\frac{dv}{dx}=\frac{x^3+(xv)^3}{x(xv)^2}}_{\boxed{\text{5 pts.}}}=\frac{x^3\left(1+v^3\right)}{x^3v}=\frac{1+v^3}{v^2}=\frac{1}{v^2}+v \Rightarrow \underbrace{x\frac{dv}{dx}=\frac{1}{v^2}}_{\boxed{\text{2 pts.}}}$$

$$\underbrace{\Rightarrow v^2\ dv=\frac{1}{x}\ dx}_{\boxed{\text{2 pts.}}} \Rightarrow \int v^2\ dv=\int \frac{1}{x}\ dx \Rightarrow \underbrace{\frac{v^3}{3}=\ln(x)+c}_{\boxed{\text{3 pts.}}} \Rightarrow \underbrace{\frac{(y/x)^3}{3}=\ln(x)+c}_{\boxed{\text{2 pts.}}} \Rightarrow$$

$(y/x)^3=3\ln(x)+\underbrace{3c}_{c_1}$. The initial condition $y(1)=3 \Rightarrow (3/1)^3=3\ln(1)+c_1 \Rightarrow c_1=27$ $\boxed{\text{1 pt.}}$.

Therefore, $(y/x)^3=3\ln(x)+27 \Rightarrow (y/x)=[3\ln(x)+27]^{1/3} \Rightarrow \boxed{\boxed{y=x\left[3\ln(x)+27\right]^{1/3}}}$.

Problem 5. (10 points) Let P denote the population of a colony of dodos. Suppose that the birth rate β (number of births per week per dodo) equals 0 and that the death rate δ (number of deaths per week per dodo) is constant. Suppose the initial population is 100, and after ten weeks the population is 50. What is the population after 43 weeks?

$\dfrac{dP}{dt}=\beta P-\delta P=(0)P-(k)\,P=-kP.$ $\boxed{\text{3 pts.}}$

This is a separable d.e: $\dfrac{dP}{dt}=-kP \Rightarrow \dfrac{dP}{P}=-k\ dt$ $\boxed{\text{1 pt.}}$

$\Rightarrow \displaystyle\int P^{-1}\ dP=\int -k\ dt \Rightarrow \ln(P)=-kt+c \Rightarrow P=e^{-kt+c}=e^{-kt}\underbrace{e^c}_{c_1}$. $\boxed{\text{3 pts.}}$

$P(0)=100 \Rightarrow 100=e^0c_1 \Rightarrow c1=100$ $\boxed{\text{1 pt.}}$

$\Rightarrow P=100e^{-kt}$. $P(10)=50 \Rightarrow 50=100e^{-10k} \Rightarrow \dfrac{50}{100}=e^{-10k} \Rightarrow \ln(0.5)=\ln\left(e^{-10k}\right)=-10k \Rightarrow k=-\dfrac{\ln(0.5)}{10}$. $\boxed{\text{1 pt.}}$

Therefore, $\boxed{\boxed{P(43)=100e^{-43k}\approx 5\text{ dodos}}}$ $\boxed{\text{1 pt.}}$.