92.236 Engineering Differential Equations Exam # 3 Solutions Spring 2015

Problem 1. (20 pts.) Solve the following differential equations.

a. (8 pts.) $y'' + 2y' + 2y = 0$

Characteristic equation: $r^2 + 2r + 2 = 0 \Rightarrow$
$r = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \dfrac{-2 \pm \sqrt{-4}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm 1i$ $\boxed{\text{4 pts.}}$

Therefore, $y = c_1 e^{-x}\cos(1x) + c_2 e^{-x}\sin(1x)$, or $\boxed{\boxed{y = c_1 e^{-x}\cos(x) + c_2 e^{-x}\sin(x)}}$ $\boxed{\text{4 pts.}}$

b. (12 pts.) $y^{(3)} - 10y'' + 25y' = 0$

Characteristic equation: $r^3 - 10r^2 + 25r = 0 \Rightarrow r\left(r^2 - 10r + 25\right) = 0 \Rightarrow r(r-5)^2 = 0 \Rightarrow$
$r = 0$ or $r = 5$ (double root). $\boxed{\text{4 pts.}}$ Therefore, $y = c_1 e^{0x} + c_2 e^{5x} + c_3 x e^{5x}$, or

$\boxed{\boxed{y = c_1 + c_2 e^{5x} + c_3 x e^{5x}}}$ $\boxed{\text{8 pts.}}$

Problem 2. (25 pts.) Solve the following initial value problem:

$$y'' + 2y' + y = 4x + 8e^x,\ y(0) = 2,\ y'(0) = -2.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' + 2y' + y = 0$.
Characteristic equation: $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$ double root.
Therefore, $y_c = c_1 e^{-x} + c_2 x e^{-x}$. $\boxed{\text{5 pts.}}$

Step 2. Find y_p.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $4x + 8e^x$ in the given d.e. is the sum of a polynomial of degree one and an exponential function, we should guess that y_p is the sum of a polynomial of degree one and an exponential function: $y_p = Ax + B + Ce^x$. $\boxed{\text{4 pts.}}$ No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{\text{2 pts.}}$ $y = Ax + B + Ce^x \Rightarrow y' = A + Ce^x \Rightarrow y'' = Ce^x$. Therefore, the left side of the d.e. is $y'' + 2y' + y = Ce^x + 2\left[A + Ce^x\right] + Ax + B + Ce^x = Ax + (B + 2A) + 4Ce^x$. We want this to equal the nonhomogeneous term $4x + 8e^x$: $Ax + (B+2A) + 4Ce^x = 4x + 8e^x \Rightarrow A = 4,\ B + 2A = 0,\ 4C = 8 \Rightarrow A = 4,\ B = -8,\ C = 2$. Thus, $y_p = 4x - 8 + 2e^x$. $\boxed{\text{9 pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-x}$ and $y_2 = xe^{-x}$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-x}\left(e^{-x} - xe^{-x}\right) - \left(-e^{-x}\right)xe^{-x} = e^{-2x}.\ \boxed{\text{1 pt.}}$$

$u_1 = \displaystyle\int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{xe^{-x}\left(4x + 8e^x\right)}{e^{-2x}}\,dx = -\int \left[4x^2e^x + 8xe^{2x}\right]\,dx =$
$-4\left[x^2e^x - 2\displaystyle\int xe^x\,dx\right] - 2\int (2x)e^{2x}\,d(2x) = -4\left[x^2e^x - 2(x-1)e^x\right] - 2(2x-1)e^{2x} =$
$\left(-4x^2 + 8x - 8\right)e^x + \left(-4x + 2\right)e^{2x}$ using formulas 46 and 47 from the table of integrals. $\boxed{\text{4 pts.}}$

$u_2 = \displaystyle\int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-x}\left(4x + 8e^x\right)}{e^{-2x}}\,dx = \int \left[4xe^x + 8e^{2x}\right]\,dx = 4\left[(x-1)e^x\right] + 4e^{2x}$ using formula 46 from the table of integrals. $\boxed{\text{4 pts.}}$

Therefore, $y_p = u_1y_1 + u_2y_2 = \left[\left(-4x^2+8x-8\right)e^x + (-4x+2)\,e^{2x}\right]e^{-x} + \left[4\left((x-1)e^x\right)+4e^{2x}\right]xe^{-x} =$ $-4x^2+8x-8+(-4x+2)\,e^x + 4(x-1)x + 4xe^x = 4x-8+2e^x$ $\boxed{\text{5 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1e^{-x} + c_2xe^{-x} + 4x - 8 + 2e^x$. $\boxed{\text{3 pts.}}$

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.
$y = c_1e^{-x} + c_2xe^{-x} + 4x - 8 + 2e^x \Rightarrow y' = -c_1e^{-x} + c_2\left[e^{-x} - xe^{-x}\right] + 4 + 2e^x$.
$y(0) = 2 \Rightarrow 2 = c_1e^0 + c_2(0)e^0 + 4(0) - 8 + 2e^0 = c_1 - 6 \Rightarrow c_1 = 8$
$y'(0) = -2 \Rightarrow -2 = -c_1e^0 + c_2\left[e^0 - (0)e^0\right] + 4 + 2e^0 = -c_1 + c_2 + 6 \Rightarrow c_2 = c_1 - 8 = 0$
$\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = 8e^{-x} + 4x - 8 + 2e^x}}$

Problem 3. (20 points) Consider a forced, damped mass-spring system with mass $m = 1$ kg, damping coefficient $c = 2$ Ns/m, spring constant $k = 16$ N/m, and external force $F_e(t) = 32\cos(4t)$ N. Find the steady periodic solution $x_{sp}(t)$.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_e(t)$. $\boxed{\text{2 pts.}}$
In this problem, the d.e. becomes $x'' + 2x' + 16x = 32\cos(4t)$. $\boxed{\text{2 pts.}}$

The steady periodic solution is the particular solution x_p. $\boxed{\text{4 pts.}}$ Since the nonhomogeneous term $32\cos(4t)$ is a cosine, we should guess that x_p is a combination of a cosine and a sine with the same frequency: $x_p = A\cos(4t) + B\sin(4t)$. $\boxed{\text{5 pts.}}$ (No part of this guess will duplicate part of x_c because x_c is a transient term containing decaying exponential functions.)
$x = A\cos(4t) + B\sin(4t) \Rightarrow x' = -4A\sin(4t) + 4B\cos(4t) \Rightarrow x'' = -16A\cos(4t) - 16B\sin(4t)$.
Therefore, the left side of the d.e. is
$x'' + 2x' + 16x = -16A\cos(4t) - 16B\sin(4t) + 2\left[-4A\sin(4t) + 4B\cos(4t)\right] + 9\left[A\cos(4t) + B\sin(4t)\right]$
$= 8B\cos(4t) - 8A\sin(4t)$.
We want this to equal the nonhomogeneous term $32\cos(4t)$:
$8B\cos(4t) - 8A\sin(4t) = 32\cos(4t) \Rightarrow 8B = 32,\ -8A = 0 \Rightarrow A = 0$ and $B = 4$. Therefore,
$\boxed{\boxed{x_{\text{sp}} = 4\sin(4t)}}$. $\boxed{\text{7 pts.}}$

Problem 4. (20 points) Solve the system $\begin{cases} x' = 2x + y \\ y' = 3x \end{cases}$

Note: $x' = dx/dt$ and $y' = dy/dt$. t is the independent variable.

Take the derivative of both sides of the second d.e. in the system: $y' = 3x \Rightarrow y'' = 3x'$. The first d.e. in the system is $x' = 2x + y$. Therefore, $y'' = 3(2x + y) = 6x + 3y$. From the second d.e. in the system, $6x = 2y'$, so we have $y'' = 2y' + 3y$ $\boxed{\text{8 pts.}}$

$y'' = 2y' + 3y \Rightarrow y'' - 2y' - 3y = 0$.
Characteristic equation: $r^2 - 2r - 3 = 0 \Rightarrow (r+1)(r-3) = 0 \Rightarrow r = -1$ or $r = 3 \Rightarrow y = c_1e^{-t} + c_2e^{3t}$.
$\boxed{\text{8 pts.}}$

The second d.e. in the given system says $x = y'/3$, so $x = \left(-c_1e^{-t} + 3c_2e^{3t}\right)/3 = -\frac{1}{3}c_1e^{-t} + c_2e^{3t}$.
Therefore, the solution of the given system is
$\boxed{\boxed{x = -\frac{1}{3}c_1e^{-t} + c_2e^{3t},\ y = c_1e^{-t} + c_2e^{3t}}}$ $\boxed{\text{4 pts.}}$