MATH.2360 Engineering Differential Equations Project Spring 2018

DUE DATES: MONDAY, MARCH 26.
OPTIONAL REVISION DUE FRIDAY, APRIL 20.

No extensions will be given except in case of illness or other emergency.

General Guidelines

1. You may work in groups of up to four. Turn in just one paper for the whole group. Each member of the group will receive the same grade. **If you would like to work in a group but cannot find a group to join, please let me know no later than Monday, February 26.**
2. If you choose to turn in a revised version of your report, 90% of your grade will be based on the final version and 10% on the first version.
3. This is a major assignment, and you should plan to do some work on it every day, starting today. **DO NOT WAIT UNTIL THE LAST MINUTE.**
4. The paper you turn in must be your group's work. I reserve the right to ask any or all of you for a verbal explanation of your solution.
5. I strongly encourage you to see me regularly as you work on the project to discuss your progress. If you receive any assistance from anyone other than me, of if you use our textbook or any other outside source, you must cite the source in your paper. Remember guideline number 4.

What Should You Hand In?

After you have completely solved all parts of the assigned problem, you should write a report explaining your solution. The report should contain a mixture of text, equations, and graphs, and possibly tables and diagrams. Your text should be grammatically correct, and you should use proper punctuation and spelling. If you do not type the paper, please write legibly. Your paper should contain an introduction explaining the problem and should clearly explain each step of your solution. **Assume the reader knows something about differential equations but has not read the project description below.** Please note that you will be graded on the quality of your presentation as well as on the mathematical content. You may find the following checklist helpful. It was adapted from a checklist developed by Dr. Annalisa Crannell of Franklin and Marshall College. Does this paper

1. clearly restate the problem to be solved?
2. clearly label diagrams, tables, and graphs?
3. define all variables used?
4. provide a paragraph explaining how the problem will be approached?
5. explain how each formula is derived or give a reference indicating where it can be found?
6. give acknowledgment where it is due?

In this paper,

7. are the spelling, grammar, and punctuation correct?
8. is the mathematics correct?
9. did you answer all the questions that were asked?

Grade Sheet

1. Derivation of d.e. (1) (5 points) ______

2a. Derivation of d.e. (2) (10 points) ______

2b (i). Calculation of γ (2 points) ______

2b (ii). Graph of numerical solution (10 points) ______

2b (iii). Long-term behavior of solution (3 points) ______

2c (i). Phase line (8 points) ______

2c (ii). Long-term behavior of solution from phase line (4 points) ______

2c (iii). Solution formula (10 points) ______

2c (iv). Long-term behavior of solution from formula (4 points) ______

3a. Derivation of d.e. (3) (5 points) ______

3b (i). Calculation of α and β (3 points) ______

3b (ii). Graph of numerical solution, analysis of long-term behavior (12 points) ______

3b (iii). Phase line, analysis of long-term behavior (14 points) ______

3c (i). Phase line (7 points – extra credit) ______

3c (ii). Long-term behavior of solution from phase line (3 points – extra credit) ______

4. Report (introduction, clarity and completeness of presentation, grammar). (10 points) ______

Total. ______

Background Information

The purpose of this project is to develop and analyze a mathematical model describing the temperature of a resistor connected to a constant voltage source. In particular, you will determine the long-term behavior of the temperature. (Does the temperature keep increasing? Does it approach a limiting value? Does it oscillate? You will find out what the model predicts.) In the heating/cooling problems you analyzed earlier in the semester using Newton's Law of Cooling, the temperature of the object changed only as a result of heat exchange with the environment. In addition to exchanging heat with the environment, resistors generate heat when current passes through them. You will need to take this into account when you develop your model.

In the remainder of this project description, the following notation will be used.

t	time (seconds)
T	temperature of the resistor at time t (K)
Q	internal energy of the resistor at time t (joule)
R	resistance of the resistor (ohms)
V	voltage across the resistor (volts), assumed constant
m	mass of the resistor (gm), assumed constant
c	specific heat of resistor (joule/(gm K)), assumed constant
k	coefficient from Newton's Law of Cooling (joule/(sec K)), assumed constant
A	room temperature (K), assumed constant
$s = kt/mc$	dimensionless time
$u = T/A$	dimensionless temperature of the resistor

We make the following assumptions about the system in order to develop the model:

- Heat exchange with the environment is described by Newton's Law of Cooling: $\dfrac{dQ}{dt} = -k\,(T - A)$.
- The power (joule/sec) consumed by the resistor is V^2/R, and the consumed energy appears as internal energy (heat) in the resistor.
- The change in internal energy of an object ΔQ is related to the temperature change ΔT of the object by the equation $\Delta Q = mc\Delta T$, where m is the mass of the object and c is the specific heat of the material from which the object is made. Therefore, the rate at which the temperature of the object changes is related to the rate of change of internal energy by $\dfrac{dT}{dt} = \dfrac{1}{mc}\dfrac{dQ}{dt}$

Here are the steps you need to carry out.

1. **Formulation of the Model.** Use the assumptions listed above to show that the temperature of the resistor satisfies the differential equation

$$\frac{dT}{dt} = \frac{1}{mc}\left[\frac{V^2}{R} - k\,(T - A)\right] \tag{1}$$

2. **Analysis of the Model Assuming Constant Resistance.** For this part of the project, assume that the resistance R is constant.

   a. Let $s = kt/(mc)$ and $u = T/A$. Show that when expressed in terms of these new variables, equation (1) becomes

   $$\frac{du}{ds} = \gamma - u \tag{2}$$

   where $\gamma = \dfrac{V^2}{kAR} + 1$.

   b. First consider the special case $V = 9.00$ volts, $m = 1.00$ gm, $c = 0.386$ joule/(gm K), $R = 20.0$ ohm, $k = 0.0193$ joule/(sec K), and $A = 293$ K.

      i. Show that $\gamma \approx 1.72$.

      ii. Use MATLAB's routine *ode45* to generate a numerical solution of equation (2) for $0 \leq s \leq 5$, taking $\gamma = 1.72$ and $u(0) = 1$. Graph your solution.

      iii. How does u behave as s increases? What does this tell you about the behavior of the temperature T of the resistor as t increases?

   c. Now consider the general case, in which parameters are not assigned specific values.

      i. Draw the phase line for equation (2).

      ii. Use your phase line to determine how u behaves as s increases. What does this tell you about the long-term behavior of T, the temperature of the resistor? Does your answer agree with the numerical prediction from part b?

      iii. Find a formula for the general solution of equation (2), either by hand or using MATLAB's *mupad* utility.

      iv. Use your solution formula from the previous step to determine how u behaves as s increases. Does your answer agree with the numerical prediction and the phase line prediction?

3. **Analysis of the Model Assuming Temperature-dependent Resistance.** For this part of the project, assume that $R = aT - b$ with $a > 0$; i.e., assume that resistance increases linearly with temperature. (This assumption is only valid for $T > b/a$.)

   a. Show that when expressed in terms of $s = kt/(mc)$ and $u = T/A$, equation (1) becomes

   $$\frac{du}{ds} = \frac{1}{\alpha u - \beta} + 1 - u \tag{3}$$

   where $\alpha = akA^2/V^2$ and $\beta = bkA/V^2$.

   b. Consider the special case $V = 9.00$ volts, $m = 1.00$ gm, $c = 0.386$ joule/(gm K), $R = 20.0$ ohm, $k = 0.0193$ joule/(sec K), $A = 293$ K, $a = 0.0695$ ohm/K, and $b = 3.56$ ohm.

      i. Show that $\alpha \approx 1.42$ and $\beta \approx 0.249$.

      ii. Use MATLAB's routine *ode45* to generate a numerical solution of equation (3) for $0 \leq s \leq 5$, taking $\alpha = 1.42$, $\beta = 0.249$, and $u(0) = 1$. Graph your solution. How does u behave as s increases? What does this tell you about the behavior of T, the temperature of the resistor?

   c. **Extra Credit** Consider the general case, in which parameters are not assigned specific values.

      i. Draw the phase line for equation (3).

      ii. Use your phase line to determine how u behaves as s increases.