Problem 1. 20 points Consider the autonomous differential equation $\frac{dx}{dt} = 2x^2 - x^3$.

a. Find all critical points (equilibrium solutions) of this d.e.

$2x^2 - x^3 = 0 \Rightarrow x^2(2-x) = 0 \Rightarrow$ $\boxed{\boxed{\text{the equilibrium solutions are } x = 0 \text{ and } x = 2}}$ $\boxed{\text{3 pts.}}$

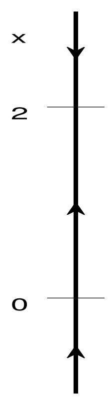
b. Draw the phase line (phase diagram) for this d.e. $\boxed{\text{8 pts.}}$

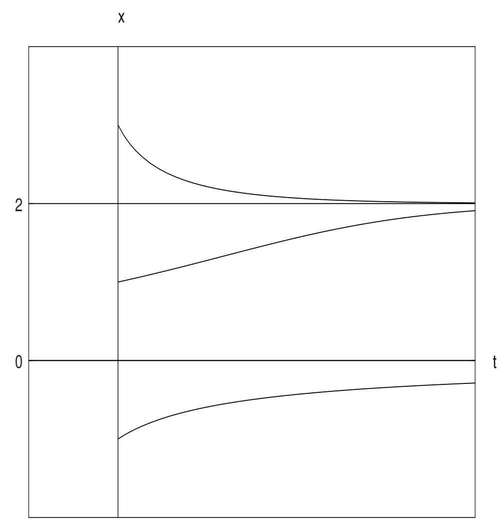
The two equilibrium solutions divide the phase line into 3 intervals: $x > 2$, $0 < x < 2$, and $x < 0$.

$\left.\frac{dx}{dt}\right|_{x=3} = 3^2(2-3) < 0$, so the direction arrow points down for $x > 2$.

$\left.\frac{dx}{dt}\right|_{x=1} = 1^2(2-1) > 0$, so the direction arrow points up for $0 < x < 2$.

$\left.\frac{dx}{dt}\right|_{x=-1} = (-1)^2(2-(-1)) > 0$, so the direction arrow points up for $x < 0$.





c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\boxed{\text{2 is stable and 0 is unstable}}}$. $\boxed{\text{2 pts.}}$

d. If $x(0) = 3$, what value will $x(t)$ approach as t increases?

Since 3 lies in the interval $x > 2$, we can see from the phase line that $\boxed{\boxed{x(t) \to 2}}$ as t increases. $\boxed{\text{3 pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. $\boxed{\text{4 pts.}}$

See the figure above.

Problem 2. (20 points) Solve the following differential equations:

a. $y'' + 6y' + 5y = 0$

Characteristic equation: $r^2 + 6r + 5 = 0 \Rightarrow (r+5)(r+1) = 0 \Rightarrow$
$r = -5$ or $r = -1$. $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-5x} + c_2 e^{-x}}}$ $\boxed{\text{6 pts.}}$

b. $y'' - 8y' + 16y = 0$

Characteristic equation: $r^2 - 8r + 16 = 0 \Rightarrow (r-4)^2 = 0 \Rightarrow$
$r = 4$ (double root). $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{4x} + c_2 x e^{4x}}}$ $\boxed{\text{6 pts.}}$

Problem 3. (20 points) Solve the following initial value problem.

$$3y^2 + \left(x^2 - 3xy\right)\frac{dy}{dx} = 0, \quad y(1) = 1$$

$3y^2 + \left(x^2 - 3xy\right)\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{3y^2}{3xy - x^2}$. dy/dx equals a rational function, and every term has the same degree (2). Therefore, this d.e. is homogeneous. $\boxed{\text{4 pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{3y^2}{3xy - x^2} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{3(xv)^2}{3x(xv) - x^2}}_{\boxed{\text{4 pts.}}} = \frac{3x^2v^2}{x^2(3v-1)} = \frac{3v^2}{3v-1} \Rightarrow$$

$$\underbrace{x\frac{dv}{dx} = \frac{3v^2}{3v-1} - v = \frac{3v^2 - v(3v-1)}{3v-1} = \frac{v}{3v-1}}_{\boxed{\text{3 pts.}}}$$

$$\underbrace{\Rightarrow \frac{3v-1}{v}\,dv = \frac{1}{x}\,dx}_{\boxed{\text{2 pts.}}} \Rightarrow \int\left(3 - \frac{1}{v}\right)dv = \int \frac{1}{x}\,dx \Rightarrow \underbrace{3v - \ln(v) = \ln(x) + c}_{\boxed{\text{3 pts.}}} \Rightarrow \underbrace{\frac{3y}{x} - \ln\left(\frac{y}{x}\right) = \ln(x) + c}_{\boxed{\text{2 pts.}}}$$

The initial condition $y(1) = 1 \Rightarrow \dfrac{3(1)}{1} - \ln\left(\dfrac{1}{1}\right) = \ln(1) + c \Rightarrow c = 3$ $\boxed{\text{2 pts.}}$.

Therefore, $\boxed{\boxed{\dfrac{3y}{x} - \ln\left(\dfrac{y}{x}\right) = \ln(x) + 3}}$

Problem 4. (20 points) Solve the following initial value problem.

$$xy^2 + 4x^3 + \left(x^2y + 2y\right)\frac{dy}{dx} = 0, \quad y(1) = 2$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{xy^2 + 4x^3}_{M} + \underbrace{\left(x^2y + 2y\right)}_{N}\frac{dy}{dx} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[xy^2 + 4x^3\right] = 2xy.$ $\boxed{\text{1 pt.}}$ $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[x^2y + 2y\right] = 2xy.$ $\boxed{\text{1 pt.}}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{3 pts.}}$ Therefore, the solution of the d.e. is $f(x,y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = xy^2 + 4x^3$ and $\dfrac{\partial f}{\partial y} = N = x^2y + 2y$.

$\dfrac{\partial f}{\partial x} = xy^2 + 4x^3 \Rightarrow f = \displaystyle\int \left(xy^2 + 4x^3\right)\, \partial x = \dfrac{x^2y^2}{2} + x^4 + g(y)$ $\boxed{\text{6 pts.}}$

$\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[\dfrac{x^2y^2}{2} + x^4 + g(y)\right] = x^2y + g'(y)$

But $\dfrac{\partial f}{\partial y} = N = x^2y + 2y \Rightarrow x^2y + g'(y) = x^2y + 2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 \Rightarrow$

$f = \dfrac{x^2y^2}{2} + x^4 + y^2$ $\boxed{\text{6 pts.}}$

Therefore, the solution of the d.e. is $\dfrac{x^2y^2}{2} + x^4 + y^2 = c$ $\boxed{\text{2 pts.}}$

$y(1) = 2 \Rightarrow \dfrac{1^2 2^2}{2} + 1^4 + 2^2 = c \Rightarrow c = 7.$ $\boxed{\text{1 pt.}}$

Therefore, the solution of the initial value problem is $\boxed{\boxed{\dfrac{x^2y^2}{2} + x^4 + y^2 = 7}}$

Problem 5. (10 points) Let P denote the population of a colony of tribbles. Suppose that the birth rate β (number of births per week per tribble) is proportional to $\sqrt{P}$ and that the death rate δ (number of deaths per week per tribble) equals 0. Suppose the initial population is 4, and after two weeks the population is 9. What is the population after 3 weeks?

$\dfrac{dP}{dt} = \beta P - \delta P = \left(k\sqrt{P}\right) P - (0)\, P = kP^{3/2}.$ $\boxed{\text{3 pts.}}$

This is a separable d.e: $\dfrac{dP}{dt} = kP^{3/2} \Rightarrow \dfrac{dP}{P^{3/2}} = k\ dt \Rightarrow \displaystyle\int P^{-3/2}\ dP = \int k\ dt \Rightarrow -2P^{-1/2} = kt + c.$

$\boxed{\text{4 pts.}}$

$P(0) = 4 \Rightarrow -2(4)^{-1/2} = k(0) + c \Rightarrow c = -1 \Rightarrow -2P^{-1/2} = kt - 1$ $\boxed{\text{1 pt.}}$

$P(2) = 9 \Rightarrow -2(9)^{-1/2} = k(2) - 1 \Rightarrow -\dfrac{2}{3} = 2k - 1 \Rightarrow k = \dfrac{1}{6}.$ $\boxed{\text{1 pt.}}$

Therefore, $-2\left(P(3)\right)^{-1/2} = \left(\dfrac{1}{6}\right)(3) - 1 = -\dfrac{1}{2} \Rightarrow \left(P(3)\right)^{-1/2} = \dfrac{1}{4} \Rightarrow \boxed{\boxed{P(3) = 16 \text{ tribbles}}}.$ $\boxed{\text{1 pt.}}$