Problem 1. (20 pts.) Solve the following differential equations.

a. (8 pts.) $y'' - 6y' + 10y = 0$

Characteristic equation: $r^2 - 6r + 10 = 0 \Rightarrow r = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)} = \dfrac{6 \pm \sqrt{-4}}{2} = \dfrac{6 \pm 2i}{2} = 3 \pm i$

$\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{3x}\cos(x) + c_2 e^{3x}\sin(x)}}$ $\boxed{\text{4 pts.}}$

b. (12 pts.) $y^{(4)} + 4y'' = 0$

Characteristic equation: $r^4 + 4r^2 = 0 \Rightarrow r^2\left(r^2+4\right) = 0 \Rightarrow r = 0$ (double root) or $r = \pm 2i$.

$\boxed{\text{4 pts.}}$ Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{0x}\cos(2x) + c_4 e^{0x}\sin(2x)$, or $\boxed{\boxed{y = c_1 + c_2 x + c_3\cos(2x) + c_4\sin(2x)}}$

$\boxed{\text{8 pts.}}$

Problem 2. (25 pts.) Solve the following initial value problem:

$$y'' - y' - 2y = 12x + 8e^{3x},\ y(0) = 5,\ y'(0) = 0.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' - y' - 2y = 0$.
Characteristic equation: $r^2 - r - 2 = 0 \Rightarrow (r+1)(r-2) = 0 \Rightarrow r = -1$ or $r = 2$.
Therefore, $y_c = c_1 e^{-x} + c_2 e^{2x}$. $\boxed{\text{5 pts.}}$

Step 2. Find y_p.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $12x + 8e^{3x}$ in the given d.e. is the sum of a polynomial of degree 1 and an exponential function, we should guess that y_p is a sum of a polynomial of degree 1 and an exponential function:
$y_p = Ax + B + Ce^{3x}$. $\boxed{\text{4 pts.}}$ No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{\text{2 pts.}}$ $y = Ax + B + Ce^{3x} \Rightarrow y' = A + 3Ce^{3x} \Rightarrow$
$y'' = 9Ce^{3x}$. Therefore, the left side of the d.e. is
$y'' - y' - 2y = 9Ce^{3x} - \left[A + 3Ce^{3x}\right] - 2\left[Ax + B + Ce^{3x}\right] = -2Ax + (-A - 2B) + 4Ce^{3x}$. We want this to equal the nonhomogeneous term $12x + 8e^{3x}$:
$-2Ax + (-A-2B) + 4Ce^{3x} = 12x + 8e^{3x} \Rightarrow -2A = 12,\ -A - 2B = 0.\ 4C = 8 \Rightarrow A = -6,\ B = 3,\ C = 2$. Thus, $y_p = -6x + 3 + 2e^{3x}$. $\boxed{\text{9 pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-x}$ and $y_2 = e^{2x}$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = e^{-x}\left(2e^{2x}\right) - \left(-e^{-x}\right)e^{2x} = 3e^{x}. \ \boxed{\text{1 pt.}}$$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{2x}\left(12x + 8e^{3x}\right)}{3e^x}\,dx = -\frac{1}{3}\int e^{x}\left[12x + 8e^{3x}\right]\,dx = -\frac{1}{3}\int\left[12xe^{x} + 8e^{4x}\right]\,dx$$

$= -\dfrac{1}{3}\left[12(x-1)e^{x} + 2e^{4x}\right]$ using formula 46 from the integral table. $\boxed{\text{4 pts.}}$

$$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-x}\left(12x + 8e^{3x}\right)}{3e^x}\,dx = \frac{1}{3}\int e^{-2x}\left[12x + 8e^{3x}\right]\,dx = \frac{1}{3}\int\left[12xe^{-2x} + 8e^{x}\right]$$

$= \dfrac{1}{3}\left[-3e^{-2x}(2x+1) + 8e^{x}\right]$ using formula 46 from the integral table, with $u = -2x$. $\boxed{\text{4 pts.}}$

Therefore, $y_p = u_1y_1 + u_2y_2 = -\frac{1}{3}\left[12(x-1)e^x + 2e^{4x}\right]e^{-x} + \frac{1}{3}\left[-3e^{-2x}(2x+1) + 8e^x\right]e^{2x} = -6x + 3 + 2e^{3x}$ $\boxed{\text{5 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1e^{-x} + c_2e^{2x} - 6x + 3 + 2e^{3x}$. $\boxed{\text{3 pts.}}$

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.

$y = c_1e^{-x} + c_2e^{2x} - 6x + 3 + 2e^{3x} \Rightarrow y' = -c_1e^{-x} + 2c_2e^{2x} - 6 + 6e^{3x}$.

$y(0) = 5 \Rightarrow 5 = c_1e^0 + c_2e^0 - 6(0) + 3 + 2e^0 = c_1 + c_2 + 5 \Rightarrow c_1 + c_2 = 0$

$y'(0) = 0 \Rightarrow 0 = -c_1e^0 + 2c_2e^0 - 6 + 6e^0 = -c_1 + 2c_2 \Rightarrow -c_1 + 2c_2 = 0.c_1 + c_2 = 0,\ -c_1 + 2c_2 = 0 \Rightarrow c_1 = 0,\ c_2 = 0$ $\boxed{\text{2 pts.}}$ Therefore, $\boxed{\boxed{y = -6x + 3 + 2e^{3x}}}$

Problem 3. (20 pts.) Consider an unforced, undamped mass-spring system with mass $m = 2$ kg, damping constant $c = 0$ N·s/m, and spring constant $k = 32$ N/m. Suppose $x(0) = -2$ m and $x'(0) = 8$ m/s.

a. Find the position $x(t)$.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_e(t)$. $\boxed{\text{3 pts.}}$
In this problem, $c = 0$ and $F_e(t) = 0$ so the d.e. becomes $2x'' + 32x = 0$. $\boxed{\text{3 pts.}}$
The characteristic equation is $2r^2 + 32 = 0$ so $r^2 = -16 \Rightarrow r = \pm 4i \Rightarrow x = c_1\cos(4t) + c_2\sin(4t)$ $\boxed{\text{10 pts.}}$

$x(0) = -2 \Rightarrow -2 = c_1\cos(0) + c_2\sin(0) = c_1$ $\boxed{\text{1 pt.}}$ so $x = -2\cos(4t) + c_2\sin(4t) \Rightarrow x' = 8\sin(4t) + 4c_2\cos(4t)$.

$x'(0) = 8 \Rightarrow 8 = 8\sin(0) + 4c_2\cos(0) = 4c_2 \Rightarrow c_2 = 2$. $\boxed{\text{1 pt.}}$ Therefore, $\boxed{\boxed{x = -2\cos(4t) + 2\sin(4t)}}$

b. Express your solution from part a in the form $x = C\cos(\omega_0 t - \alpha)$

$x = c_1\cos(4t) + c_2\sin(4t)$ where $c_1 = -2$ and $c_2 = 2$. $C = \sqrt{c_1^2 + c_2^2} = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$ $\boxed{\text{1 pt.}}$

Because $c_1 < 0$ we have $\alpha = \pi + \tan^{-1}(c_2/c_1) = \pi + \tan^{-1}(2/-2)) = \pi + \tan^{-1}(-1) = \pi - \pi/4 = 3\pi/4$.

$\boxed{\text{1 pt.}}$ Therefore, $\boxed{\boxed{x = 2\sqrt{2}\cos(4t - 3\pi/4)}}$

Problem 4. (20 points) Consider an RLC circuit with inductance $H = 1$ henry, resistance $R = 3\Omega$, capacitance $C = 0.5$ farad, and applied voltage $E(t) = 20\cos(2t)$ volts. Find the steady periodic current $I_{sp}(t)$.

The d.e. describing an RLC circuit is $LQ'' + RQ' + \frac{Q}{C} = E(t)$. $\boxed{\text{2 pts.}}$
In this problem, the d.e. becomes $Q'' + 3Q' + 2Q = 20\cos(2t)$. $\boxed{\text{1 pt.}}$

The steady periodic solution is the particular solution. $\boxed{\text{3 pts.}}$ Since the nonhomogeneous term $20\cos(2t)$ is a cosine, we should guess that Q_p is a combination of a cosine and a sine with the same frequency: $Q_p = A\cos(2t) + B\sin(2t)$. $\boxed{\text{5 pts.}}$ (No part of this guess will duplicate part of Q_c because Q_c is a transient term containing decaying exponential functions.)

$Q = A\cos(2t) + B\sin(2t) \Rightarrow Q' = -2A\sin(2t) + 2B\cos(2t) \Rightarrow$

$Q'' = -4A\cos(2t) - 4B\sin(2t)$. Therefore, the left side of the d.e. is

$Q'' + 3Q' + 2Q = -4A\cos(2t) - 4B\sin(2t) + 3\left[-2A\sin(2t) + 2B\cos(2t)\right] + 2\left[A\cos(2t) + B\sin(2t)\right]$

$= (-4A + 6B + 2A)\cos(2t) + (-4B - 6A + 2B)\sin(2t) = (-2A + 6B)\cos(2t) + (-6A - 2B)\sin(2t)$.

We want this to equal the nonhomogeneous term $20\cos(2t)$:

$(-2A + 6B)\cos(2t) + (-6A - 2B)\sin(2t) = 20\cos(2t) \Rightarrow -2A + 6B = 20,\ -6A - 2B = 0 \Rightarrow$

$A = -1$ and $B = 3$. Therefore, $Q_{\text{sp}} = -\cos(2t) + 3\sin(2t)$. $\boxed{\text{8 pts.}}$ Current is the derivative of Q, so $\boxed{\boxed{I_{\text{sp}} = 2\sin(2t) + 6\cos(2t)}}$ $\boxed{\text{1 pt.}}$