92.236 Engineering Differential Equations Exam # 1 Solutions Spring 2018

Problem 1. (10 points)

Is $y(x) = x^3$ is a solution of the d.e. $yy' = x^5 + y$? Why or why not?

Left side of d.e: $y = x^3 \Rightarrow y' = 3x^2 \Rightarrow yy' = \left(x^3\right)\left(3x^2\right) = 3x^5$. $\boxed{\text{4 pts.}}$

Right side of d.e: $y = x^3 \Rightarrow x^5 + y = x^5 + x^3$. $\boxed{\text{3 pts.}}$

Left side $\neq$ right side, so $\boxed{\boxed{y(x) = x^3 \textbf{ is not} \text{ a solution of the d.e. } yy' = x^5 + y}}$ $\boxed{\text{3 pts.}}$

Problem 2. (15 points)

A ball is dropped from rest (initial velocity 0) from a height of 20 meters. How long does it take the ball to reach the ground? Use the value 10 m/s^2 for g, the acceleration due to gravity.

For objects moving in one dimension under constant acceleration, the velocity and position of the object are given by the formulas $v = at + v_0$ and $x = \frac{1}{2}at^2 + v_0 t + x_0$. $\boxed{\text{5 pts.}}$ Take $t = 0$ to be the time the ball is dropped, take $x = 0$ to be ground level, and take up to be the positive x direction. Then $a = -g$, $x_0 = 20$ m, and $v_0 = 0$ m/s, so $x = -\frac{1}{2}gt^2 + 20$. $\boxed{\text{5 pts.}}$ Set $x = 0$ and solve for t to find the time when the ball hits the ground: $0 = -\frac{1}{2}gt^2 + 20 \Rightarrow t = \sqrt{40/g} \approx \sqrt{40/10} =$ $\boxed{\boxed{\text{2 seconds}}}$. $\boxed{\text{5 pts.}}$

Problem 3. (25 points)

Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{2x^2 + y}{x}, \quad y(1) = 3.$$

$\frac{dy}{dx} = \frac{2x^2 + y}{x} \Rightarrow \frac{dy}{dx} = 2x + \frac{y}{x}$This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone. $\boxed{\text{5 pts.}}$

First write the equation in standard form:

$\frac{dy}{dx} = 2x + \frac{y}{x} \Rightarrow \frac{dy}{dx} - \left(\frac{1}{x}\right) y = 2x$ $\boxed{\text{3 pts.}}$

Next, find the integrating factor: $\rho(x) = e^{\int -1/x \, dx} = e^{-\ln(x)} = x^{-1}$. $\boxed{\text{6 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^{-1}\left[\frac{dy}{dx} - \left(\frac{1}{x}\right) y\right] = x^{-1}(2x) \Rightarrow x^{-1}\frac{dy}{dx} - x^{-2}y = 2.$ $\boxed{\text{2 pts.}}$

Use the Product Rule backwards to rewrite the d.e. as $\frac{d}{dx}\left[x^{-1}y\right] = 2.$ $\boxed{\text{4 pts.}}$

Integrating both sides, we obtain $x^{-1}y = \int 2 \, dx = 2x + c.$ $\boxed{\text{3 pts.}}$

$y(1) = 3 \Rightarrow 1^{-1}(3) = 2(1) + c \Rightarrow c = 1$ $\boxed{\text{2 pts.}}$

Therefore, $x^{-1}y = 2x+1$, so $\boxed{\boxed{y = 2x^2 + x.}}$

Problem 4. (25 points)

Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{2xy}{y^2+1}, \quad y(0) = 1.$$

This is a separable d.e. $\boxed{\text{5 pts.}}$

$\frac{dy}{dx} = \frac{2xy}{y^2+1} \Rightarrow dy = \left(\frac{2xy}{y^2+1}\right)\, dx \Rightarrow \frac{y^2+1}{y}\, dy = 2x\, dx.$ $\boxed{\text{5 pts.}}$

$\Rightarrow \int \frac{y^2+1}{y}\, dy = \int 2x\, dx \Rightarrow \int \left(y + \frac{1}{y}\right)\, dy = \int 2x\, dx \Rightarrow \frac{y^2}{2} + \ln(y) = x^2 + c.$ $\boxed{\text{12 pts.}}$

$y(0) = 1 \Rightarrow \frac{1^2}{2} + \ln(1) = 0^2 + c \Rightarrow c = \frac{1}{2}$ $\boxed{\text{3 pts.}}$

$\Rightarrow \frac{y^2}{2} + \ln(y) = x^2 + \frac{1}{2}$ or $\boxed{\boxed{y^2 + 2\ln(y) = 2x^2 + 1}}$.

Problem 5. (15 points)

A tank initially contains 100 liters of water in which 200 grams of salt are dissolved. A salt solution containing 20 grams of salt per liter is pumped into the tank at the rate of 5 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 3 liters per minute.

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in grams). Write down the differential equation $\left(\frac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.

$\frac{dx}{dt}$ = rate in - rate out $\boxed{\text{3 pts.}}$

= (flow rate in)(concentration in) - (flow rate out)(concentration out), $\boxed{\text{3 pts.}}$ so

$$\frac{dx}{dt} = \underbrace{\left(5\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(20\frac{\text{gm}}{\text{L}}\right)}_{\boxed{\text{1 pt.}}} - \underbrace{\left(3\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}}\underbrace{\left(\frac{x\text{ gm}}{(100+2t)\text{ L}}\right)}_{\boxed{\text{5 pts.}}}.$$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 100 + (5-3)t$ liters.)

Initially there are 200 gm. of salt in the tank, so $x(0) = 200$ $\boxed{\text{1 pt.}}$.

Therefore, the initial value problem describing this mixing problem is $\boxed{\boxed{\frac{dx}{dt} = 100 - \frac{3x}{100+2t} \quad \text{with } x(0) = 200.}}$