MATH.2360 Engineering Differential Equations
Solutions to Sample Problems for Exam # 2

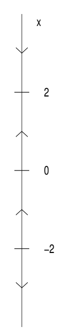
Problem 1. Consider the autonomous differential equation $\frac{dx}{dt} = 4x^2 - x^4$.

a. Find all critical points (equilibrium solutions) of this d.e.

$4x^2 - x^4 = 0 \Rightarrow x^2\left(4 - x^2\right) = 0 \Rightarrow x^2(2+x)(2-x) = 0 \Rightarrow$
$\boxed{\text{the critical points are } -2, 0 \text{ and } 2}$

b. Draw the phase line (phase diagram) for this d.e.

The three critical points divide the phase line into 4 intervals: $x > 2,\ 0 < x < 2,\ -2 < x < 0,$ and $x < -2$.
$\left.\frac{dx}{dt}\right|_{x=3} = 3^2(2+3)(2-3) < 0$, so the direction arrow points down for $x > 2$.
$\left.\frac{dx}{dt}\right|_{x=1} = 1^2(2+1)(2-1) > 0$, so the direction arrow points up for $0 < x < 2$.
$\left.\frac{dx}{dt}\right|_{x=-1} = (-1)^2(2-1)(2-(-1)) > 0$, so the arrow points up for $-2 < x < 0$.
$\left.\frac{dx}{dt}\right|_{x=-3} = (-3)^2(2-3)(2-(-3)) < 0$, so the arrow points down for $x < -2$.



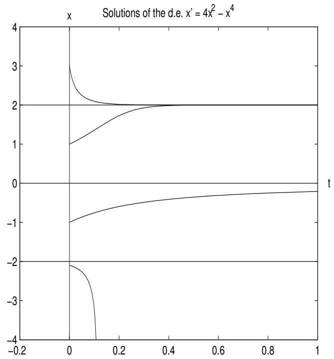
c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\text{2 is stable but } -2 \text{ and } 0 \text{ are unstable}}$.

d. If $x(0) = -1$, what value will $x(t)$ approach as t increases?

Since -1 lies in the interval $-2 < x < 0$, we can see from the phase line that $\boxed{x(t) \to 0}$ as t increases.

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes.



Problem 2. Solve the following initial value problem.

$$3x^2 + 6xy + \left[3x^2 + 2y\right] \frac{dy}{dx} = 0, \quad y(1) = 0.$$

$$\underbrace{3x^2 + 6xy}_{M} + \underbrace{\left[3x^2 + 2y\right]}_{N} \frac{dy}{dx} = 0.$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[3x^2 + 6xy\right] = 6x$. $\quad \dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[3x^2 + 2y\right] = 6x$.
Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. Therefore, the solution of the d.e. is $f(x, y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 3x^2 + 6xy$ and $\dfrac{\partial f}{\partial y} = N = 3x^2 + 2y$. $\dfrac{\partial f}{\partial x} = 3x^2 + 6xy \Rightarrow$
$f = \displaystyle\int \left(3x^2 + 6xy\right)\, dx = x^3 + 3x^2y + g(y) \Rightarrow \dfrac{\partial f}{\partial y} = 3x^2 + g'(y)$.
But $\dfrac{\partial f}{\partial y} = N = 3x^2 + 2y$, so $3x^2 + g'(y) = 3x^2 + 2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2$.
Therefore, $f = x^3 + 3x^2y + y^2$, so the solution of the d.e. is $x^3 + 3x^2y + y^2 = c$.
The initial condition $y(1) = 0 \Rightarrow 1^3 + 3(1)^2(0) + 0^2 = c \Rightarrow c = 1$. Therefore, the solution of the given IVP is $\boxed{\boxed{x^3 + 3x^2y + y^2 = 1}}$

Problem 3. Solve the following initial value problem.

$$x^3 + y^3 - xy^2 \frac{dy}{dx} = 0, \quad y(1) = 3.$$

$x^3 + y^3 - xy^2 \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}$. Since dy/dx equals a rational function in which each term has the same degree (3), the d.e. is homogeneous. We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2} \Rightarrow v + x\frac{dv}{dx} = \frac{x^3+(xv)^3}{x(xv)^2} = \frac{x^3\left(1+v^3\right)}{x^3 v} = \frac{1+v^3}{v^2} = \frac{1}{v^2} + v \Rightarrow x\frac{dv}{dx} = \frac{1}{v^2}$$
$$\Rightarrow v^2\ dv = \frac{1}{x}\ dx \Rightarrow \int v^2\ dv = \int \frac{1}{x}\ dx \Rightarrow \frac{v^3}{3} = \ln(x) + c \Rightarrow \frac{(y/x)^3}{3} = \ln(x) + c \Rightarrow$$

$(y/x)^3 = 3\ln(x) + \underbrace{3c}_{c_1}$. The initial condition $y(1) = 3 \Rightarrow (3/1)^3 = 3\ln(1) + c_1 \Rightarrow c_1 = 27$ $\boxed{\text{1 pt.}}$.

Therefore, $(y/x)^3 = 3\ln(x) + 27 \Rightarrow (y/x) = [3\ln(x)+27]^{1/3} \Rightarrow$ $\boxed{\boxed{y = x\,[3\ln(x)+27]^{1/3}}}$.

Problem 4. Solve the following initial value problem.

$$x^4 + 4y^4 - 4xy^3\frac{dy}{dx} = 0, \quad y(1) = 2.$$

$x^4 + 4y^4 - 4xy^3\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{x^4+4y^4}{4xy^3}$. dy/dx equals a rational function, and every term has the same degree (4). Therefore, this d.e. is homogeneous.

We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{x^4+4y^4}{4xy^3} \Rightarrow v + x\frac{dv}{dx} = \frac{x^4+4(xv)^4}{4x(xv)^3} = \frac{x^4\left(1+4v^4\right)}{4x^4v^3} = \frac{1+4v^4}{4v^3} = \frac{1}{4v^3} + v \Rightarrow x\frac{dv}{dx} = \frac{1}{4v^3}$$
$$\Rightarrow 4v^3\ dv = \frac{1}{x}\ dx \Rightarrow \int 4v^3\ dv = \int \frac{1}{x}\ dx \Rightarrow v^4 = \ln(x) + c \Rightarrow \left(\frac{y}{x}\right)^4 = \ln(x) + c$$

The initial condition $y(1) = 2 \Rightarrow (2/1)^4 = \ln(1) + c \Rightarrow c = 16$.

Therefore, $\left(\dfrac{y}{x}\right)^4 = \ln(x) + 16 \Rightarrow \dfrac{y}{x} = [\ln(x)+16]^{1/4} \Rightarrow$ $\boxed{\boxed{y = x\,[\ln(x)+16]^{1/4}}}$.

Problem 5. (20 points) Solve the following initial value problem.

$$2xy^2 + 3x^2 + \left(2x^2y + 4y^3\right)\frac{dy}{dx} = 0, \quad y(1) = 2.$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{2xy^2 + 3x^2}_{M} + \underbrace{\left(2x^2y + 4y^3\right)}_{N}\frac{dy}{dx} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[2xy^2 + 3x^2\right] = 4xy$. $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[2x^2y + 4y^3\right] = 4xy$.

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. Therefore, the solution of the d.e. is $f(x,y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 2xy^2 + 3x^2$ and $\dfrac{\partial f}{\partial y} = N = 2x^2y + 4y^3$.

$$\frac{\partial f}{\partial x} = 2xy^2 + 3x^2 \Rightarrow f = \int \left(2xy^2 + 3x^2\right)\ \partial x = x^2y^2 + x^3 + g(y)$$
$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2y^2 + x^3 + g(y)\right] = 2x^2y + g'(y)$$

But $\dfrac{\partial f}{\partial y} = N = 2x^2y + 4y^3 \Rightarrow 2x^2y + g'(y) = 2x^2y + 4y^3 \Rightarrow g'(y) = 4y^3 \Rightarrow g(y) = y^4 \Rightarrow$
$f = x^2y^2 + x^3 + y^4$

Therefore, the solution of the d.e. is $x^2y^2 + x^3 + y^4 = c$
$y(1) = 2 \Rightarrow 1^2 2^2 + 1^3 + 2^4 = c \Rightarrow c = 21.$

Therefore, the solution of the initial value problem is $\boxed{\boxed{x^2y^2 + x^3 + y^4 = 21}}$

Problem 6. (10 points) Let P denote the population of a colony of dodos. Suppose that the birth rate β (number of births per week per dodo) equals 0 and that the death rate δ (number of deaths per week per dodo) is constant. Suppose the initial population is 100, and after ten weeks the population is 50. What is the population after 43 weeks?

$$\frac{dP}{dt} = \beta P - \delta P = (0)P - (k)\,P = -kP.$$

This is a separable d.e: $\dfrac{dP}{dt} = -kP \Rightarrow \dfrac{dP}{P} = -k\ dt$

$$\Rightarrow \int P^{-1}\ dP = \int -k\ dt \Rightarrow \ln(P) = -kt + c \Rightarrow P = e^{-kt+c} = e^{-kt}\underbrace{e^c}_{c_1}.$$

$P(0) = 100 \Rightarrow 100 = e^0 c_1 \Rightarrow c_1 = 100$

$$\Rightarrow P = 100e^{-kt}.\ P(10) = 50 \Rightarrow 50 = 100e^{-10k} \Rightarrow \frac{50}{100} = e^{-10k} \Rightarrow \ln(0.5) = \ln\left(e^{-10k}\right) = -10k \Rightarrow k = -\frac{\ln(0.5)}{10}.$$

Therefore, $\boxed{\boxed{P(43) = 100e^{-43k} \approx 5 \text{ dodos}}}$.

Problem 7. (10 points) A ball of mass m falling vertically downward experiences two forces: its weight, and a drag force proportional to the *square* of its velocity. Let t denote time, and let v denote the ball's velocity at time t. (Assume $v < 0$ for the falling object. In other words, up is the positive direction.)

Write a differential equation (dv/dt = something) modeling the ball's motion. Make sure the drag force has the proper sign (+ or −). All parameters are assumed to be positive.

DO NOT SOLVE THE DIFFERENTIAL EQUATION, JUST WRITE IT DOWN.

Newton's Second Law says that $F = ma$, where F is the total force acting on an object and a is the object's acceleration.

The total force acting on the ball is $F = \underbrace{-mg}_{\text{weight}} + \underbrace{kv^2}_{\text{drag}}$

(Because the ball is falling, the drag is in the upward direction, so the drag force must be positive.)

Therefore, $F = ma \Rightarrow -mg + kv^2 = m\dfrac{dv}{dt} \Rightarrow \dfrac{dv}{dt} = -g + \underbrace{\dfrac{k}{m}}_{k_1} v^2 \Rightarrow \boxed{\boxed{\dfrac{dv}{dt} = -g + k_1 v^2}}$