

MATH.2360 Engineering Differential Equations  
Solutions to Sample Problems for Exam # 3

**Problem 1.** Solve the following differential equations.

a.  $y'' + 2y' + 2y = 0$ .

Characteristic equation:  $r^2 + 2r + 2 = 0 \Rightarrow$

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm 1i$$

Therefore,  $y = c_1e^{-x} \cos(1x) + c_2e^{-x} \sin(1x)$ , or  $y = c_1e^{-x} \cos(x) + c_2e^{-x} \sin(x)$

b.  $y'' - 10y' + 25y = 0$

Characteristic equation:  $r^2 - 10r + 25 = 0 \Rightarrow (r - 5)^2 = 0 \Rightarrow r = 5$  (double root)

Therefore,  $y = c_1e^{5x} + c_2xe^{5x}$

c.  $y^{(3)} - 10y'' + 25y' = 0$ .

Characteristic equation:  $r^3 - 10r^2 + 25r = 0 \Rightarrow r(r^2 - 10r + 25) = 0 \Rightarrow r(r - 5)^2 = 0 \Rightarrow r = 0$  or  $r = 5$  (double root). Therefore,  $y = c_1e^{0x} + c_2e^{5x} + c_3xe^{5x}$ , or

$y = c_1 + c_2e^{5x} + c_3xe^{5x}$

d.  $y'' + y' - 2y = 0$

Characteristic equation:  $r^2 + r - 2 = 0 \Rightarrow (r + 2)(r - 1) = 0 \Rightarrow r = -2$  or  $r = 1$ .

Therefore,  $y = c_1e^{-2x} + c_2e^x$

e.  $y'' - 6y' + 25y = 0$ .

Characteristic equation:  $r^2 - 6r + 25 = 0 \Rightarrow$

$$r = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(25)}}{2(1)} = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$$

Therefore,  $y = c_1e^{3x} \cos(4x) + c_2e^{3x} \sin(4x)$

f.  $y^{(4)} + 4y^{(3)} + 4y'' = 0$ .

Characteristic equation:  $r^4 + 4r^3 + 4r^2 = 0 \Rightarrow r^2(r^2 + 4r + 4) = 0 \Rightarrow r^2(r + 2)^2 = 0 \Rightarrow r = 0$  or  $r = -2$  (both double roots). Therefore,  $y = c_1e^{0x} + c_2xe^{0x} + c_3e^{-2x} + c_4xe^{-2x}$ , or

$y = c_1 + c_2x + c_3e^{-2x} + c_4xe^{-2x}$

**Problem 2.** Solve the following initial value problem:

$$y'' + 2y' + y = 4x + 8e^x, \quad y(0) = 2, \quad y'(0) = -2.$$

Step 1. Find  $y_c$  by solving the homogeneous d.e.  $y'' + 2y' + y = 0$ .

Characteristic equation:  $r^2 + 2r + 1 = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1$  double root.

Therefore,  $y_c = c_1e^{-x} + c_2xe^{-x}$ .

Step 2. Find  $y_p$ .

Method 1: Undetermined Coefficients. Since the nonhomogeneous term  $4x + 8e^x$  in the given d.e. is the sum of a polynomial of degree one and an exponential function, we should guess that  $y_p$  is the sum of a polynomial of degree one and an exponential function:  $y_p = Ax + B + Ce^x$ . No term in this guess duplicates a term in  $y_c$ , so there is no need to modify this guess.  $y = Ax + B + Ce^x \Rightarrow y' = A + Ce^x \Rightarrow y'' = Ce^x$ . Therefore, the left side of the d.e. is  $y'' + 2y' + y = Ce^x + 2[A + Ce^x] + Ax + B + Ce^x = Ax + (B + 2A) + 4Ce^x$ . We want this to equal the nonhomogeneous term  $4x + 8e^x$ :  $Ax + (B + 2A) + 4Ce^x = 4x + 8e^x \Rightarrow A = 4, B + 2A = 0, 4C = 8 \Rightarrow A = 4, B = -8, C = 2$ . Thus,  $y_p = 4x - 8 + 2e^x$ .

Method 2: Variation of Parameters. From  $y_c$  we obtain two independent solutions of the homogeneous d.e:  $y_1 = e^{-x}$  and  $y_2 = xe^{-x}$ . The Wronskian is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-x}(e^{-x} - xe^{-x}) - (-e^{-x})xe^{-x} = e^{-2x}.$$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)} dx = - \int \frac{xe^{-x}(4x + 8e^x)}{e^{-2x}} dx = - \int [4x^2e^x + 8xe^{2x}] dx = -4 \left[ x^2e^x - 2 \int xe^x dx \right] - 2 \int (2x)e^{2x} d(2x) = -4 \left[ x^2e^x - 2(x-1)e^x \right] - 2(2x-1)e^{2x} = (-4x^2 + 8x - 8)e^x + (-4x + 2)e^{2x} \text{ using formulas 46 and 47 from the table of integrals.}$$

$$u_2 = \int \frac{y_1 f(x)}{W(x)} dx = \int \frac{e^{-x}(4x + 8e^x)}{e^{-2x}} dx = \int [4xe^x + 8e^{2x}] dx = 4[(x-1)e^x] + 4e^{2x} \text{ using formula 46 from the table of integrals.}$$

$$\text{Therefore, } y_p = u_1y_1 + u_2y_2 = [(-4x^2 + 8x - 8)e^x + (-4x + 2)e^{2x}]e^{-x} + [4((x-1)e^x) + 4e^{2x}]xe^{-x} = -4x^2 + 8x - 8 + (-4x + 2)e^x + 4(x-1)x + 4xe^x = 4x - 8 + 2e^x$$

Step 3.  $y = y_c + y_p$ , so  $y = c_1e^{-x} + c_2xe^{-x} + 4x - 8 + 2e^x$ .

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.

$$y = c_1e^{-x} + c_2xe^{-x} + 4x - 8 + 2e^x \Rightarrow y' = -c_1e^{-x} + c_2[e^{-x} - xe^{-x}] + 4 + 2e^x.$$

$$y(0) = 2 \Rightarrow 2 = c_1e^0 + c_2(0)e^0 + 4(0) - 8 + 2e^0 = c_1 - 6 \Rightarrow c_1 = 8$$

$$y'(0) = -2 \Rightarrow -2 = -c_1e^0 + c_2[e^0 - (0)e^0] + 4 + 2e^0 = -c_1 + c_2 + 6 \Rightarrow c_2 = c_1 - 8 = 0$$

Therefore, 
$$\boxed{y = 8e^{-x} + 4x - 8 + 2e^x}$$

**Problem 3.** Solve the following initial value problem:

$$y'' - 2y' = 15 \sin(x), \quad y(0) = 0, \quad y'(0) = -3.$$

Step 1. Find  $y_c$  by solving the homogeneous d.e.  $y'' - 2y' = 0$ .

Characteristic equation:  $r^2 - 2r = 0 \Rightarrow r(r - 2) = 0 \Rightarrow r = 0$  or  $r = 2$ .

Therefore,  $y_c = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$ .

Step 2. Find  $y_p$ .

Method 1: Undetermined Coefficients. Since the nonhomogeneous term  $15 \sin(x)$  in the given d.e. is a sine function, we should guess that  $y_p$  is a combination of a cosine function and a sine function with the same coefficient of  $x$  as in the nonhomogeneous term:  $y_p = A \cos(x) + B \sin(x)$ . No term in this guess duplicates a term in  $y_c$ , so there is no need to modify the guess.  $y = A \cos(x) + B \sin(x) \Rightarrow y' = -A \sin(x) + B \cos(x) \Rightarrow y'' = -A \cos(x) - B \sin(x)$ . Therefore, the left side of the d.e. is  $y'' - 2y' = -A \cos(x) - B \sin(x) - 2[-A \sin(x) + B \cos(x)] = [-A - 2B] \cos(x) + [2A - B] \sin(x)$ . We want this to equal the nonhomogeneous term  $15 \sin(x)$  so  $2A - B = 15$ ,  $-A - 2B = 0 \Rightarrow A = 6$ ,  $B = -3$ . Thus,  $y_p = 6 \cos(x) - 3 \sin(x)$ .

Method 2: Variation of Parameters. From  $y_c$  we obtain two independent solutions of the homogeneous d.e:  $y_1 = 1$  and  $y_2 = e^{2x}$ . The Wronskian is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = (1)(2e^{2x}) - (0)e^{-2x} = 2e^{2x}.$$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)} dx = - \int \frac{e^{2x} (15 \sin(x))}{2e^{2x}} dx = -\frac{15}{2} \int \sin(x) dx = \frac{15}{2} \cos(x)$$

$$u_2 = \int \frac{y_1 f(x)}{W(x)} dx = \int \frac{(1)(15 \cos(x))}{2e^{2x}} dx = \frac{15}{2} \int [e^{-2x} \cos(x)] dx = \frac{15}{2} \left\{ \frac{e^{-2x}}{(-2)^2 + 1^2} [-2 \sin(x) - \cos(x)] \right\}$$

$$= -\frac{15}{2} e^{-2x} [2 \sin(x) + \cos(x)] \text{ using formula 49 from the Table of Integrals.}$$

$$\text{Therefore, } y_p = u_1 y_1 + u_2 y_2 = \left[ \frac{15}{2} \cos(x) \right] (1) + \left\{ -\frac{3}{2} e^{-2x} [2 \sin(x) + \cos(x)] \right\} e^{2x} =$$

$$\frac{15}{2} \cos(x) - 3 \sin(x) - \frac{3}{2} \cos(x) = 6 \cos(x) - 3 \sin(x).$$

Step 3.  $y = y_c + y_p$ , so  $y = c_1 + c_2 e^{2x} + 6 \cos(x) - 3 \sin(x)$ .

Step 4. Use the initial conditions to determine the value of  $c_1$  and  $c_2$ .

$$y = c_1 + c_2 e^{2x} + 6 \cos(x) - 3 \sin(x) \Rightarrow y' = 2c_2 e^{2x} - 6 \sin(x) - 3 \cos(x).$$

$$y(0) = 0 \Rightarrow 0 = c_1 + c_2 e^0 + 6 \cos(0) - 3 \sin(0) = c_1 + c_2 + 6 \Rightarrow c_1 + c_2 = -6.$$

$$y'(0) = -3 \Rightarrow -3 = 2c_2 e^0 - 6 \sin(0) - 3 \cos(0) = 2c_2 - 3 \Rightarrow c_2 = 0. \quad c_1 + c_2 = -6 \Rightarrow c_1 = -6 - c_2 = -6.$$

$$\text{Therefore, } \boxed{y = -6 + 6 \cos(x) - 3 \sin(x)}$$

**Problem 4.** Consider an RLC circuit with inductance  $L = 1$  henry, resistance  $R = 2\Omega$ , capacitance  $C = 1/16$  farad, and applied voltage  $E(t) = 32 \cos(4t)$  volts. Find the steady periodic current  $I_{sp}(t)$ .

The d.e. describing an RLC circuit is  $LQ'' + RQ' + \frac{Q}{C} = E(t)$ .

In this problem, the d.e. becomes  $Q'' + 2Q' + 16Q = 32 \cos(4t)$ .

The steady periodic solution is the particular solution. Since the nonhomogeneous term  $32 \cos(4t)$  is a cosine, we should guess that  $Q_p$  is a combination of a cosine and a sine with

the same frequency:  $Q_p = A \cos(4t) + B \sin(4t)$ . (No part of this guess will duplicate part of  $Q_c$  because  $Q_c$  is a transient term containing decaying exponential functions.)

$$Q = A \cos(4t) + B \sin(4t) \Rightarrow Q' = -4A \sin(4t) + 4B \cos(4t) \Rightarrow$$

$Q'' = -16A \cos(4t) - 16B \sin(4t)$ . Therefore, the left side of the d.e. is

$$Q'' + 2Q' + 16Q = -16A \cos(4t) - 16B \sin(4t) + 2[-4A \sin(4t) + 4B \cos(4t)] + 16[A \cos(4t) + B \sin(4t)] \\ = 8B \cos(4t) - 8A \sin(4t).$$

We want this to equal the nonhomogeneous term  $32 \cos(4t)$ :

$$8B \cos(4t) - 8A \sin(4t) = 32 \cos(4t) \Rightarrow 8B = 32, \quad -8A = 0 \Rightarrow A = 0 \text{ and } B = 4. \text{ Therefore,}$$

$$Q_{sp} = 4 \sin(4t). \text{ Current is the derivative of } Q, \text{ so } \boxed{I_{sp} = 16 \cos(4t)}$$

**Problem 5.** Consider a free (unforced), damped mass-spring system with mass  $m = 1$  kg, damping constant  $c = 6$  N·s/m, and spring constant  $k = 10$  N/m. Assume that  $x(0) = -1$  and  $x'(0) = 0$ .

a. Find the position function  $x(t)$ .

The d.e. describing a mass-spring system is  $mx'' + cx' + kx = F_e(t)$ .

In this problem, the d.e. becomes  $x'' + 6x' + 10x = 0$ .

The characteristic equation is  $r^2 + 6r + 10 = 0 \Rightarrow$

$$r = \frac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)} = \frac{-6 \pm \sqrt{-4}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i$$

Therefore,  $x = c_1 e^{-3t} \cos(t) + c_2 e^{-3t} \sin(t)$  so  $x' = c_1 [-3e^{-t} \cos(t) - e^{-3t} \sin(t)] + c_2 [-3e^{-t} \sin(t) + e^{-3t} \cos(t)]$

$$x(0) = -1 \Rightarrow -1 = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = c_1 \Rightarrow c_1 = -1.$$

$$x'(0) = 0 \Rightarrow 0 = c_1 [-3e^0 \cos(0) - e^0 \sin(0)] + c_2 [-3e^0 \sin(0) + e^0 \cos(0)] = -3c_1 + c_2 \Rightarrow c_2 = 3c_1 = -3$$

$$\text{Therefore } \boxed{x = -e^{-3t} \cos(t) - 3e^{-3t} \sin(t)}$$

b. Express your solution from part a in the form  $x = C e^{-pt} \cos(\omega_1 t - \alpha)$

$$C = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$$

Because  $c_1 < 0$  we have  $\alpha = \pi + \tan^{-1}(c_2/c_1) = \pi + \tan^{-1}(3)$

$$\text{Therefore, } \boxed{x = \sqrt{10} e^{-3t} \cos\left(t - \left(\pi + \tan^{-1}(3)\right)\right)}$$

c. Is this system overdamped, underdamped, or critically damped?

The characteristic equation has complex roots, so  $\boxed{\text{the system is underdamped.}}$