# 92.236 Engineering Differential Equations    Exam # 1 Solutions    Spring 2019

**Problem 1. (10 points)**

Is $y(x) = x^2$ is a solution of the d.e. $xy' + x^4 = y^2$? Why or why not?

Left side of d.e: $y = x^2 \Rightarrow y' = 2x \Rightarrow xy' + x^4 = x(2x) + x^4 = 2x^2 + x^4$. $\boxed{\text{4 pts.}}$

Right side of d.e: $y = x^2 \Rightarrow y^2 = \left(x^2\right)^2 = x^4$. $\boxed{\text{3 pts.}}$

Left side $\neq$ right side, so $\boxed{\boxed{y(x) = x^2 \textbf{ is not } \text{a solution of the d.e. } xy' + x^4 = y^2}}$ $\boxed{\text{3 pts.}}$

**Problem 2. (15 points)**

A can of soda at a temperature of 45°F is brought into a room where the temperature is 75°F. After 10 minutes the temperature of the can of soda is 50°F. What will the temperature of the soda be after 30 minutes?

Let $t$ denote time in minutes, let $T$ denote the temperature of the can of soda in °F, and let $A$ denote room temperature °F. As we showed in class, $T = A + (T_0 - A)\,e^{-kt}$ where $T_0 = T(0)$. Here $T_0 = 45$°F amd $A = 75$°F, so $T = 75 + (45 - 75)\,e^{-kt} = 75 - 30e^{-kt}$. $\boxed{\text{6 pts.}}$

$T(10) = 50 \Rightarrow 50 = 75 - 30e^{-k(10)} \Rightarrow 50 - 75 = -30e^{-10k} \Rightarrow \dfrac{-25}{-30} = e^{-10k} \Rightarrow$

$\ln(5/6) = \ln\left(e^{-10k}\right) = -10k \Rightarrow k = -\dfrac{\ln(5/6)}{10}$ $\boxed{\text{6 pts.}}$

Therefore, $T(30) = 75 - 30e^{-k(30)} = 75 - 30e^{30(\ln(5/6)/10)} \Rightarrow \boxed{\boxed{T(30) = 75 - 30e^{3\ln(5/6)} \approx 57.6°\text{F}}}$ $\boxed{\text{3 pts.}}$

**Problem 3. (25 points)**

Solve the following initial value problem:

$$x^2\frac{dy}{dx} = 2xy + 6, \quad y(1) = 3.$$

This is a linear d.e. because $y$ and $dy/dx$ appear just to the first power, multiplied by functions of $x$ alone. $\boxed{\text{5 pts.}}$

First write the equation in standard form:

$x^2\dfrac{dy}{dx} = 2xy + 6 \Rightarrow x^2\dfrac{dy}{dx} - 2xy = 6 \Rightarrow \dfrac{dy}{dx} - \left(\dfrac{2}{x}\right)y = \dfrac{6}{x^2}$ $\boxed{\text{3 pts.}}$

Next, find the integrating factor: $\rho(x) = e^{\int -2/x\ dx} = e^{-2\ln(x)} = x^{-2}$. $\boxed{\text{6 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^{-2}\left[\dfrac{dy}{dx} - \left(\dfrac{2}{x}\right)y\right] = x^{-2}\left(\dfrac{6}{x^2}\right) \Rightarrow x^{-2}\dfrac{dy}{dx} - 2x^{-3}y = 6x^{-4}$. $\boxed{\text{2 pts.}}$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\left[x^{-2}y\right] = 6x^{-4}$. $\boxed{\text{4 pts.}}$

Integrating both sides, we obtain $x^{-2}y = \displaystyle\int 6x^{-4}\ dx = -2x^{-3} + c$. $\boxed{\text{3 pts.}}$

$y(1) = 3 \Rightarrow 1^{-2}(3) = -2(1)^{-3} + c \Rightarrow c = 5$ [2 pts.]

Therefore, $x^{-2}y = -2x^{-3} + 5$, so $\boxed{y = 5x^2 - 2x^{-1}.}$

**Problem 4. (25 points)**

Solve the following initial value problem.

$$x^2 \frac{dy}{dx} = 2xy^2 \quad \text{with} \quad y(1) = 1$$

This is a separable d.e. [5 pts.]

$x^2 \frac{dy}{dx} = 2xy^2 \Rightarrow \frac{dy}{y^2} = \left(\frac{2x}{x^2}\right)\, dx \Rightarrow y^{-2}\, dy = \left(\frac{2}{x}\right)\, dx.$ [5 pts.]

$\Rightarrow \int y^{-2}\, dy = \int \left(\frac{2}{x}\right)\, dx \Rightarrow -y^{-1} = 2\ln(x) + c.$ [12 pts.]

$y(1) = 1 \Rightarrow -1^{-1} = 2\ln(1) + c \Rightarrow c = -1$ [3 pts.]

$\Rightarrow -y^{-1} = 2\ln(x) - 1 \Rightarrow y^{-1} = 1 - 2\ln(x) \Rightarrow \boxed{y = \frac{1}{1 - 2\ln(x)}}.$

**Problem 5. (15 points)**

A tank initially contains 300 liters of water in which 500 grams of salt are dissolved. A salt solution containing 10 grams of salt per liter is pumped into the tank at the rate of 5 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 3 liters per minute.

Let $t$ denote time (in minutes), and let $x$ denote the amount of salt in the tank at time $t$ (in grams). Write down the differential equation $\left(\frac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

**DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.**

$\frac{dx}{dt} = \text{rate in - rate out}$ [3 pts.]

$=$ (flow rate in)(concentration in) - (flow rate out)(concentration out), [3 pts.] so

$$\frac{dx}{dt} = \underbrace{\left(5\frac{\text{L}}{\text{min}}\right)}_{\boxed{1\text{ pt.}}} \underbrace{\left(10\frac{\text{gm}}{\text{L}}\right)}_{\boxed{1\text{ pt.}}} - \underbrace{\left(3\frac{\text{L}}{\text{min}}\right)}_{\boxed{1\text{ pt.}}} \underbrace{\left(\frac{x\text{ gm}}{(300 + 2t)\text{ L}}\right)}_{\boxed{5\text{ pts.}}}.$$

(The volume in the tank at time $t$ is initial volume + $t$ (flow rate in - flow rate out) $= 300 + (5-3)t$ liters.)

Initially there are 500 gm. of salt in the tank, so $x(0) = 500$ [1 pt.].

Therefore, the initial value problem describing this mixing problem is $\boxed{\frac{dx}{dt} = 50 - \frac{3x}{300 + 2t} \quad \text{with } x(0) = 500.}$