92.236 Engineering Differential Equations   Exam # 2 Solutions   Spring 2018

Problem 1. (20 points) Consider the autonomous differential equation $\dfrac{dx}{dt} = x(x^2 - 4x + 4)$.

a. Find all critical points (equilibrium solutions) of this d.e.

$x(x^2 - 4x + 4) = 0 \Rightarrow x(x-2)^2 = 0 \Rightarrow$ $\boxed{\boxed{\text{the equilibrium solutions are } x = 0 \text{ and } x = 2}}$ $\boxed{\text{3 pts.}}$

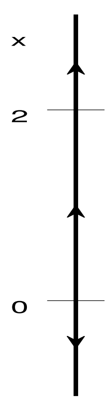
b. Draw the phase line (phase diagram) for this d.e. $\boxed{\text{8 pts.}}$

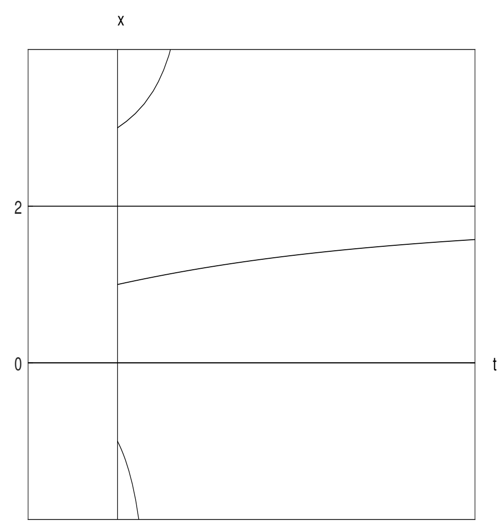
The two equilibrium solutions divide the phase line into 3 intervals: $x > 2,\ 0 < x < 2$, and $x < 0$.

$\left.\dfrac{dx}{dt}\right|_{x=3} = 3(3-2)^2 > 0$, so the direction arrow points up for $x > 2$.

$\left.\dfrac{dx}{dt}\right|_{x=1} = 1(1-2)^2 > 0$, so the direction arrow points up for $0 < x < 2$.

$\left.\dfrac{dx}{dt}\right|_{x=-1} = (-1)(-1-2)^2 < 0$, so the direction arrow points down for $x < 0$.





c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\boxed{\text{both 2 and 0 are unstable}}}$. $\boxed{\text{2 pts.}}$

d. If $x(0) = 1$, what value will $x(t)$ approach as t increases?

Since 3 lies in the interval $0 < x < 2$, we can see from the phase line that $\boxed{\boxed{x(t) \to 2}}$ as t increases. $\boxed{\text{3 pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. $\boxed{\text{4 pts.}}$

See the figure above.

Problem 2. (20 points) Solve the following initial value problem.

$$x^2 - 3y^2 + 2xy\frac{dy}{dx} = 0, \quad y(1) = 2$$

$x^2 - 3y^2 + 2xy\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$. dy/dx equals a rational function, and every term has the same degree (2). Therefore, this d.e. is homogeneous. $\boxed{\text{4 pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{3(xv)^2 - x^2}{2x(xv)}}_{\boxed{\text{4 pts.}}} = \frac{x^2\left(3v^2 - 1\right)}{2x^2 v} = \frac{3v^2 - 1}{2v} \Rightarrow$$

$$\underbrace{x\frac{dv}{dx} = \frac{3v^2 - 1}{2v} - v = \frac{3v^2 - 1 - 2v^2}{2v} = \frac{v^2 - 1}{2v}}_{\boxed{\text{3 pts.}}}$$

$$\underbrace{\Rightarrow \frac{2v}{v^2 - 1}\, dv = \frac{1}{x}\, dx}_{\boxed{\text{2 pts.}}} \Rightarrow \int \frac{2v}{v^2 - 1}\, dv = \int \frac{1}{x}\, dx \Rightarrow \underbrace{\ln\left(v^2 - 1\right) = \ln(x) + c}_{\boxed{\text{3 pts.}}} \Rightarrow \underbrace{\ln\left((y/x)^2 - 1\right) = \ln(x) + c}_{\boxed{\text{2 pts.}}}$$

The initial condition $y(1) = 2 \Rightarrow \ln\left((2/1)^2 - 1\right) = \ln(1) + c \Rightarrow c = \ln(3)$ $\boxed{\text{2 pts.}}$.

Therefore, $\boxed{\boxed{\ln\left((y/x)^2 - 1\right) = \ln(x) + \ln(3)}}$ or $\boxed{\boxed{y = x\sqrt{3x + 1}}}$

Problem 3. (20 points) Solve the following initial value problem.

$$6x + y^2 + (2xy + 2y)\frac{dy}{dx} = 0, \quad y(1) = 3$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{6x + y^2}_{M} + \underbrace{(2xy + 2y)}_{N}\frac{dy}{dx} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[6x + y^2\right] = 2y.$ $\boxed{\text{1 pt.}}$ $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[2xy + 2y\right] = 2y.$ $\boxed{\text{1 pt.}}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{3 pts.}}$ Therefore, the solution of the d.e. is $f(x, y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 6x + y^2$ and $\dfrac{\partial f}{\partial y} = N = 2xy + 2y$.

$\dfrac{\partial f}{\partial x} = 6x + y^2 \Rightarrow f = \int \left(6x + y^2\right)\, \partial x = 3x^2 + y^2x + g(y)$ $\boxed{\text{6 pts.}}$

$\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[3x^2 + y^2x + g(y)\right] = 2yx + g'(y)$

But $\dfrac{\partial f}{\partial y} = N = 2xy + 2y \Rightarrow 2yx + g'(y) = 2xy + 2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 \Rightarrow f = 3x^2 + y^2x + y^2$

$\boxed{\text{6 pts.}}$

Therefore, the solution of the d.e. is $3x^2 + y^2x + y^2 = c$ $\boxed{\text{2 pts.}}$

$y(1) = 3 \Rightarrow 3(1)^2 + 3^2(1) + 3^2 = c \Rightarrow c = 21.$ $\boxed{\text{1 pt.}}$

Therefore, the solution of the initial value problem is $\boxed{\boxed{3x^2 + y^2x + y^2 = 21}}$ or $\boxed{\boxed{y = \sqrt{\dfrac{21 - 3x^2}{x + 1}}}}$

Problem 4. (20 points) Let P denote the population of a colony of tribbles. Suppose that the birth rate β (number of births per week per tribble) is proportional to P^2 and that the death rate δ (number of deaths per week per tribble) equals 0. Suppose the initial population is 2, and after one week the population is 4. When will the population equal 8?

$\dfrac{dP}{dt} = \beta P - \delta P = \left(kP^2\right) P - (0)\, P = kP^3.$ $\boxed{\text{6 pts.}}$

This is a separable d.e: $\dfrac{dP}{dt} = kP^3 \Rightarrow \dfrac{dP}{P^3} = k\ dt \Rightarrow \int P^{-3}\ dP = \int k\ dt \Rightarrow \dfrac{P^{-2}}{-2} = kt + c.$ $\boxed{\text{8 pts.}}$

$P(0) = 2 \Rightarrow \dfrac{2^{-2}}{-2} = k(0) + c \Rightarrow c = -1/8 \Rightarrow \dfrac{P^{-2}}{-2} = kt - 1/8$ $\boxed{\text{2 pt.}}$

$P(1) = 4 \Rightarrow \dfrac{4^{-2}}{-2} = k(1) - 1/8 \Rightarrow k = 1/8 - 1/32 = 3/32 \Rightarrow \dfrac{P^{-2}}{-2} = 3t/32 - 1/8.$ $\boxed{\text{2 pt.}}$

Therefore, $P(t) = 8 \Rightarrow \dfrac{8^{-2}}{-2} = 3t/32 - 1/8 = 3t/32 = 1/8 - 1/128 = 15/128 \Rightarrow \boxed{\boxed{t = 5/4 \text{ weeks}}}$. $\boxed{\text{2 pt.}}$

Problem 5. (10 points) A car experiences two forces: a constant thrust force T and a drag force proportional to its velocity. Let t denote time, let v denote the car's velocity at time t, and let m denote the car's mass.

Write a differential equation ($dv/dt =$ something) modeling the car's motion.

DO NOT SOLVE THE DIFFERENTIAL EQUATION, JUST WRITE IT DOWN.

Newton's Second Law says that $F = ma$, where F is the total force acting on an object and a is the object's acceleration. $\boxed{\text{3 pts.}}$

The total force acting on the ball is $F = \underbrace{T}_{\text{thrust}} + \underbrace{-kv}_{\text{drag}}$

Therefore, $F = ma \Rightarrow T - kv = m\dfrac{dv}{dt} \Rightarrow \boxed{\boxed{\dfrac{dv}{dt} = \dfrac{T}{m} - \left(\dfrac{k}{m}\right) v}}$ $\boxed{\text{7 pts.}}$