92.236 Engineering Differential Equations   Exam # 2 Solutions   Spring 2019

Problem 1. (20 points) Consider the autonomous differential equation $\dfrac{dx}{dt} = x(x^2+4x+4)$.

a. Find all critical points (equilibrium solutions) of this d.e.

$x(x^2+4x+4) = 0 \Rightarrow x(x+2)^2 = 0 \Rightarrow$ $\boxed{\boxed{\text{the equilibrium solutions are } x = -2 \text{ and } x = 0}}$ $\boxed{\text{3 pts.}}$

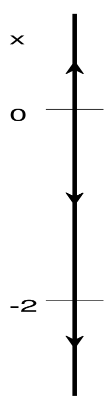
b. Draw the phase line (phase diagram) for this d.e. $\boxed{\text{8 pts.}}$

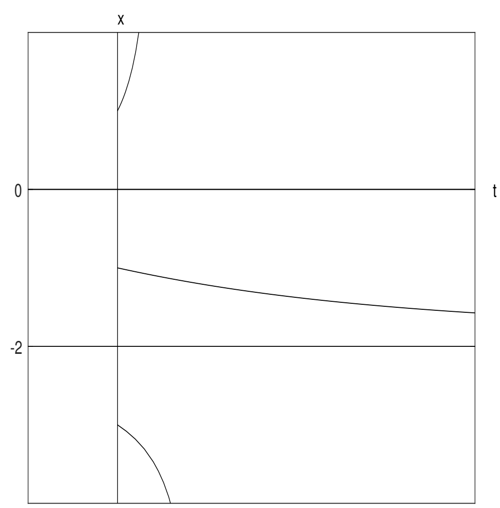
The two equilibrium solutions divide the phase line into 3 intervals: $x > 0,\ -2 < x < 0$, and $x < -2$.

$\left.\dfrac{dx}{dt}\right|_{x=1} = 1(1+2)^2 > 0$, so the direction arrow points up for $x > 0$.

$\left.\dfrac{dx}{dt}\right|_{x=-1} = (-1)(-1+2)^2 < 0$, so the direction arrow points down for $-2 < x < 0$.

$\left.\dfrac{dx}{dt}\right|_{x=-3} = (-3)(-3+2)^2 < 0$, so the direction arrow points down for $x < -2$.





c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\boxed{\text{both } -2 \text{ and } 0 \text{ are unstable}}}$. $\boxed{\text{2 pts.}}$

d. If $x(0) = -1$, what value will $x(t)$ approach as t increases?

Since 3 lies in the interval $-2 < x < 0$, we can see from the phase line that $\boxed{\boxed{x(t) \to -2}}$ as t increases. $\boxed{\text{3 pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. $\boxed{\text{4 pts.}}$

See the figure above.

Problem 2. (20 points) Solve the following initial value problem.

$$3x^2 - 6y^2 + (6y - 12xy)\frac{dy}{dx} = 0, \quad y(2) = 1$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{3x^2 - 6y^2}_{M} + \underbrace{(6y - 12xy)}_{N}\frac{dy}{dx} = 0$$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[3x^2 - 6y^2\right] = -12y.$ $\boxed{\text{1 pt.}}$ $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[6y - 12xy\right] = -12y.$ $\boxed{\text{1 pt.}}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{3 pts.}}$ Therefore, the solution of the d.e. is $f(x, y) = c$, where the function f satisfies the conditions $\frac{\partial f}{\partial x} = M = 3x^2 - 6y^2$ and $\frac{\partial f}{\partial y} = N = 6y - 12xy$.

$\frac{\partial f}{\partial x} = 3x^2 - 6y^2 \Rightarrow f = \int \left(3x^2 - 6y^2\right)\, \partial x = x^3 - 6y^2 x + g(y)$ $\boxed{\text{6 pts.}}$

$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^3 - 6y^2 x + g(y)\right] = -12yx + g'(y)$

But $\frac{\partial f}{\partial y} = N = 6y - 12xy \Rightarrow -12yx + g'(y) = 6y - 12xy \Rightarrow g'(y) = 6y \Rightarrow g(y) = 3y^2 \Rightarrow f = x^3 - 6y^2 x + 3y^2$

$\boxed{\text{6 pts.}}$

Therefore, the solution of the d.e. is $x^3 - 6xy^2 + 3y^2 = c$ $\boxed{\text{2 pts.}}$

$y(2) = 1 \Rightarrow 2^3 - 12(1)(2)^2 + 3(2)^2 = c \Rightarrow c = -1.$ $\boxed{\text{1 pt.}}$

Therefore, the solution of the initial value problem is $\boxed{\boxed{x^3 - 6xy^2 + 3y^2 = -1}}$ or $\boxed{\boxed{y = \sqrt{\frac{x^3 + 1}{6x - 3}}}}$

Problem 3. (20 points) Solve the following initial value problem.

$$x^2 - 6y^2 + 4xy\frac{dy}{dx} = 0, \quad y(1) = 2$$

$x^2 - 6y^2 + 4xy\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{6y^2 - x^2}{4xy}$. dy/dx equals a rational function, and every term has the same degree (2). Therefore, this d.e. is homogeneous. $\boxed{\text{4 pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\frac{dy}{dx}$ by $v + x\frac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{6y^2 - x^2}{4xy} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{6(xv)^2 - x^2}{4x(xv)}}_{\boxed{4 \text{ pts.}}} = \frac{x^2(6v^2-1)}{4x^2v} = \frac{6v^2-1}{4v} \Rightarrow$$

$$\underbrace{x\frac{dv}{dx} = \frac{6v^2-1}{4v} - v = \frac{6v^2 - 1 - 4v^2}{4v} = \frac{2v^2-1}{4v}}_{\boxed{3 \text{ pts.}}}$$

$$\underbrace{\Rightarrow \frac{4v}{2v^2-1}\,dv = \frac{1}{x}\,dx}_{\boxed{2 \text{ pts.}}} \Rightarrow \int \frac{4v}{2v^2-1}\,dv = \int \frac{1}{x}\,dx \Rightarrow \underbrace{\ln\left(2v^2-1\right) = \ln(x) + c}_{\boxed{3 \text{ pts.}}} \Rightarrow \underbrace{\ln\left(2(y/x)^2 - 1\right) = \ln(x) + c}_{\boxed{2 \text{ pts.}}}$$

The initial condition $y(1) = 2 \Rightarrow \ln\left(2(2/1)^2 - 1\right) = \ln(1) + c \Rightarrow c = \ln(7)$ $\boxed{2 \text{ pts.}}$.

Therefore, $\boxed{\boxed{\ln\left(2(y/x)^2 - 1\right) = \ln(x) + \ln(7)}}$ or $\boxed{\boxed{y = x\sqrt{\frac{7x+1}{2}}}}$

Problem 4. (20 points) Let P denote the population of a colony of tribbles. Suppose that the birth rate β (number of births per week per tribble) is proportional to $\sqrt{P}$ and that the death rate δ (number of deaths per week per tribble) equals 0. Suppose the initial population is 4, and after one week the population is 9. When will the population equal 25?

$$\frac{dP}{dt} = \beta P - \delta P = \left(k\sqrt{P}\right)P - (0)\,P = kP^{3/2}. \quad \boxed{6 \text{ pts.}}$$

This is a separable d.e: $\frac{dP}{dt} = kP^{3/2} \Rightarrow \frac{dP}{P^{3/2}} = k\ dt \Rightarrow \int P^{-3/2}\ dP = \int k\ dt \Rightarrow -2P^{-1/2} = kt + c.$
$\boxed{8 \text{ pts.}}$

$P(0) = 4 \Rightarrow -2(4)^{-1/2} = k(0) + c \Rightarrow c = -1 \Rightarrow -2P^{-1/2} = kt - 1$ $\boxed{2 \text{ pt.}}$

$P(1) = 9 \Rightarrow -2(9)^{-1/2} = k(1) - 1 \Rightarrow k = 1/3 \Rightarrow -2P^{-1/2} = t/3 - 1 \Rightarrow 2P^{-1/2} = 1 - t/3.$ $\boxed{2 \text{ pt.}}$

Therefore, $P(t) = 25 \Rightarrow 2(25)^{-1/2} = 1 - t/3 \Longrightarrow 2/5 = 1 - t/3$ $\boxed{\boxed{t = 9/5 \text{ weeks}}}$. $\boxed{2 \text{ pt.}}$

Problem 5. (10 points) A car experiences two forces: a constant thrust force T and a drag force proportional to its velocity. Let t denote time, let v denote the car's velocity at time t, and let m denote the car's mass.

Write a differential equation ($dv/dt =$ something) modeling the car's motion.

DO NOT SOLVE THE DIFFERENTIAL EQUATION, JUST WRITE IT DOWN.

Newton's Second Law says that $F = ma$, where F is the total force acting on an object and a is the object's acceleration. $\boxed{3 \text{ pts.}}$

The total force acting on the ball is $F = \underbrace{T}_{\text{thrust}} + \underbrace{-kv}_{\text{drag}}$

Therefore, $F = ma \Rightarrow T - kv = m\frac{dv}{dt} \Rightarrow$ $\boxed{\boxed{\frac{dv}{dt} = \frac{T}{m} - \left(\frac{k}{m}\right)v}}$ $\boxed{7 \text{ pts.}}$