Problem 1. (20 pts.) Solve the following differential equations.

a. (10 pts.) $y'' + 6y' + 9y = 0$

Characteristic equation: $r^2 + 6r + 9 = 0 \Rightarrow (r+3)^2 = 0 \Rightarrow r = -3$ (double root) $\boxed{\text{5 pts.}}$

Therefore, $\boxed{\boxed{y = c_1 e^{-3x} + c_2 x e^{-3x}}}$ $\boxed{\text{5 pts.}}$

b. (10 pts.) $y'' - 4y = 0$.

Characteristic equation: $r^2 - 4 = 0 \Rightarrow (r+2)(r-2) = 0 \Rightarrow r = -2$ or $r = 2$ $\boxed{\text{5 pts.}}$

Therefore, $\boxed{\boxed{y = c_1 e^{-2x} + c_2 e^{2x}}}$ $\boxed{\text{5 pts.}}$

Problem 2. (20 points) Solve the following differential equations:

a. (10 pts.) $y^{(3)} + 2y'' + 5y' = 0$

Characteristic equation: $r^3 + 2r^2 + 5r = 0 \Rightarrow r\left(r^2 + 2r + 5\right) = 0 \Rightarrow$
$r = 0$ or $r = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i.$ $\boxed{\text{5 pts.}}$
Therefore, $y = c_1 e^{0x} + c_2 e^{-1x}\cos(2x) + c_3 e^{-1x}\sin(2x)$, or
$\boxed{\boxed{y = c_1 + c_2 e^{-x}\cos(2x) + c_3 e^{-x}\sin(2x)}}$ $\boxed{\text{5 pts.}}$

b. (10 pts.) $y^{(4)} - y^{(3)} - 6y'' = 0$.

Characteristic equation: $r^4 - r^3 - 6r^2 = 0 \Rightarrow r^2\left(r^2 - r - 6\right) = 0 \Rightarrow r^2(r+2)(r-3) = 0$
$\Rightarrow r = 0$ (double root) or $r = -2$ or $r = 3$. $\boxed{\text{5 pts.}}$

Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{-2x} + c_4 e^{3x}$, or $\boxed{\boxed{y = c_1 + c_2 x + c_3 e^{-2x} + c_4 e^{3x}}}$ $\boxed{\text{5 pts.}}$

Problem 3. (25 pts.) Solve the following initial value problem:

$$y'' + 4y = 8x^2,\ y(0) = 1,\ y'(0) = 0.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' + 4y = 0$.
Characteristic equation: $r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm\sqrt{-4} = 0 \pm 2i$.
Therefore, $y_c = c_1 e^{0x}\cos(2x) + c_2 e^{0x}\sin(2x) = c_1\cos(2x) + c_2\sin(2x)$. $\boxed{\text{5 pts.}}$

Step 2. Find y_p.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $8x^2$ in the given d.e. is a polynomial of degree 2, we should guess that y_p is a polynomial of degree 2:
$y_p = Ax^2 + Bx + C$. $\boxed{\text{4 pts.}}$ No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{\text{2 pts.}}$
$y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A$. Therefore, the left side of the d.e. is
$y'' + 4y = 2A + 4\left[Ax^2 + Bx + C\right] = 4Ax^2 + 4Bx + (2A + 4C)$. We want this to equal the nonhomogeneous term $8x^2$:

$4Ax^2 + 4Bx + (2A + 4C) = 8x^2 \Rightarrow 4A = 8,\ 4B = 0,\ 2A + 4C = 0 \Rightarrow A = 2,\ B = 0,\ C = -1.$
Thus, $y_p = 2x^2 - 1$. $\boxed{\text{9 pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = \cos(2x)\,(2\cos(2x)) - (-2\sin(2x))\sin(2x) =$
$2\left[\cos^2(2x) + \sin^2(2x)\right] = 2.$ $\boxed{\text{1 pt.}}$

$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{\sin(2x)\,(8x^2)}{2}\,dx = -4\int \left[x^2\sin(2x)\right]\,dx =$
$\left(2x^2 - 1\right)\cos(2x) - 2x\sin(2x)$ after 2 integrations by parts or after using formulas 38 and 40 from the table of integrals. $\boxed{\text{4 pts.}}$

$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{\cos(2x)\,(8x^2)}{2}\,dx = 4\int \left[x^2\cos(2x)\right]\,dx = \left(2x^2 - 1\right)\sin(2x) + 2x\sin(2x)$ after 2 integrations by parts or after using formulas 39 and 41 from the table of integrals. $\boxed{\text{4 pts.}}$

Therefore, $y_p = u_1y_1 + u_2y_2 =$
$\left[\left(2x^2 - 1\right)\cos(2x) - 2x\sin(2x)\right]\cos(2x) + \left[\left(2x^2 - 1\right)\sin(2x) + 2x\sin(2x)\right]\sin(2x) =$
$\left(2x^2 - 1\right)\cos^2(2x) - 2x\sin(2x)\cos(2x) + \left(2x^2 - 1\right)\sin^2(2x) + 2x\sin(2x)\cos(2x) =$
$\left(2x^2 - 1\right)\left[\cos^2(2x) + \sin^2(2x)\right] = 2x^2 - 1$ $\boxed{\text{5 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1\cos(2x) + c_2\sin(2x) + 2x^2 - 1$. $\boxed{\text{3 pts.}}$

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.
$y = c_1\cos(2x) + c_2\sin(2x) + 2x^2 - 1 \Rightarrow y' = -2c_1\sin(2x) + 2c_2\cos(2x) + 4x.$
$y(0) = 1 \Rightarrow 1 = c_1\cos(0) + c_2\sin(0) + 2(0)^2 - 1 = c_1 - 1 \Rightarrow c_1 = 2$
$y'(0) = 0 \Rightarrow 0 = -2c_1\sin(0) + 2c_2\cos(0) + 4(0) = 2c_2 \Rightarrow c_2 = 0$
$\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = 2\cos(2x) + 2x^2 - 1}}$

Problem 4. (20 points) Consider a forced, damped mass-spring system with mass $m = 1$ kg, damping constant $c = 10$ N·s/m, spring constant $k = 9$ N/m, and external force $F_{\text{ext}} = 60\cos(3t)$ N. Find the steady-state (steady periodic) solution x_{sp}.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_{\text{e}}(t)$. $\boxed{\text{2 pts.}}$
In this problem, the d.e. becomes $x'' + 10x' + 9x = 60\cos(3t)$. $\boxed{\text{2 pts.}}$

The steady periodic solution is the particular solution x_p. $\boxed{\text{4 pts.}}$

Using the Method of Undetermined Coefficients: Since the nonhomogeneous term $60\cos(3t)$ is a cosine, we should guess that x_p is a combination of a cosine and a sine with the same frequency: $x_p = A\cos(3t) + B\sin(3t)$. $\boxed{\text{5 pts.}}$ (No part of this guess will duplicate part of x_c because x_c is a transient term containing decaying exponential functions.)
$x = A\cos(3t) + B\sin(3t) \Rightarrow x' = -3A\sin(3t) + 3B\cos(3t) \Rightarrow x'' = -9A\cos(3t) - 9B\sin(3t)$. Therefore, the left side of the d.e. is
$x'' + 10x' + 9x = -9A\cos(3t) - 9B\sin(3t) + 10\left[-3A\sin(3t) + 3B\cos(3t))\right] + 9\left[A\cos(3t) + B\sin(3t)\right]$
$= 30B\cos(2t) - 30A\sin(2t).$
We want this to equal the nonhomogeneous term $60\cos(3t)$:
$30B\cos(2t) - 30A\sin(2t) = 60\cos(3t) \Rightarrow 30B = 60,\ -30A = 0 \Rightarrow A = 0$ and $B = 2$. Therefore,
$\boxed{\boxed{x_{\text{sp}} = 2\sin(3t)}}$. $\boxed{\text{7 pts.}}$

Using the Method of Variation of Parameters: We first find the complementary solution by solving the homogeneous d.e. $x'' + 10x' + 9x = 0$.

Characteristic equation: $r^2 + 10r + 9 = 0 \Rightarrow (r+9)(r+1) = 0 \Rightarrow r = -9$ or $r = -1$. Therefore, $x_c = c_1 e^{-9t} + c_2 e^{-t}$ From x_c we obtain two independent solutions of the homogeneous d.e: $x_1 = e^{-9t}$ and $x_2 = e^{-t}$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$W(t) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-9t} & e^{-t} \\ -9e^{-9t} & -e^{-t} \end{vmatrix} = e^{-9t}\left(-e^{-t}\right) - \left(-9e^{-9t}\right)e^{-t} =$
$8e^{-10t}$. $\boxed{\text{1 pt.}}$

$u_1 = \displaystyle\int \frac{-x_2 f(t)}{W(t)}\, dx = -\int \frac{e^{-t}\left(60\cos(3t)\right)}{8e^{-10t}}\, dx = -\frac{15}{2}\int \left[e^{9t}\cos(3t)\right]\, dx =$
$-\dfrac{15}{2}\left(\dfrac{e^{9t}}{9^2+3^2}\right)[9\cos(3t) + 3\sin(3t)] = -\dfrac{1}{4}e^{9t}[3\cos(3t) + \sin(3t)]$ using formula 50 from the table of integrals. $\boxed{\text{4 pts.}}$

$u_2 = \displaystyle\int \frac{x_1 f(t)}{W(t)}\, dx = \frac{15}{2}\int \left[e^{t}\cos(3t)\right]\, dx =$
$\dfrac{15}{2}\left(\dfrac{e^{t}}{1^2+3^2}\right)[\cos(3t) + 3\sin(3t)] = \dfrac{3}{4}e^{t}[\cos(3t) + 3\sin(3t)]$ again using formula 50 from the table of integrals. $\boxed{\text{4 pts.}}$

Therefore, $x_p = u_1 x_1 + u_2 x_2 =$
$\left\{-\dfrac{1}{4}e^{9t}[3\cos(3t) + \sin(3t)]\right\}e^{-9t} + \left\{\dfrac{3}{4}e^{t}[\cos(3t) + 3\sin(3t)]\right\}e^{-t} =$
$-\dfrac{3}{4}\cos(3t) - \dfrac{1}{4}\sin(3t) + \dfrac{3}{4}\cos(3t) + \dfrac{9}{4}\sin(3t) = 2\sin(3t)$ Therefore, $\boxed{\boxed{x_{\text{sp}} = 2\sin(3t)}}$. $\boxed{\text{2 pts.}}$