MATH.2360 Example Illustrating the Method of Variation of Parameters

Solve the d.e. $y'' - 3y' + 2y = 10\sin(2x)$.

Step 1. Find the complementary solution, y_c, by solving the homogeneous d.e. $y'' - 3y' + 2y = 0$. The characteristic equation is $r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0 \Rightarrow r = 1$ or $r = 2 \Rightarrow y_c = c_1 \underbrace{e^x}_{y_1} + c_2 \underbrace{e^{2x}}_{y_2}$.

Step 2. Find the particular solution, y_p. According to the Method of Variation of Parameters, $y_p = u_1 y_1 + u_2 y_2$. Here $u_1' = -y_2 f(x)/W$ and $u_2' = y_1 f(x)/W$, where

$f(x) = 10\sin(2x)$ is the right side of the given d.e. and

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x\left(2e^{2x}\right) - e^x\left(e^{2x}\right) = e^{3x}.$$

$$u_1' = -\frac{y_2 f(x)}{W} = -\frac{e^{2x}\left(10\sin(2x)\right)}{e^{3x}} = -10e^{-x}\sin(2x) \Rightarrow$$

$$u_1 = -10\int e^{-x}\sin(2x)\,dx = -10\underbrace{\frac{e^{-x}}{(-1)^2+2^2}\left[-\sin(2x) - 2\cos(2x)\right]}_{\text{from integral table}} = 2e^{-x}\left[\sin(2x) + 2\cos(2x)\right].$$

$$u_2' = \frac{y_1 f(x)}{W} = -\frac{e^x\left(10\sin(2x)\right)}{e^{3x}} = 10e^{-2x}\sin(2x) \Rightarrow$$

$$u_2 = 10\int e^{-2x}\sin(2x)\,dx = 10\underbrace{\frac{e^{-2x}}{(-2)^2+2^2}\left[-2\sin(2x) - 2\cos(2x)\right]}_{\text{from integral table}} = -\frac{5}{2}e^{-2x}\left[\sin(2x) + \cos(2x)\right].$$

$$y_p = u_1 y_1 + u_2 y_2 = \left\{2e^{-x}\left[\sin(2x) + 2\cos(2x)\right]\right\}e^x + \left\{-\frac{5}{2}e^{-2x}\left[\sin(2x) + \cos(2x)\right]\right\}e^{2x} =$$

$$2\sin(2x) + 4\cos(2x) - \frac{5}{2}\sin(2x) - \frac{5}{2}\cos(2x) = -\frac{1}{2}\sin(2x) + \frac{3}{2}\cos(2x).$$

Step 3. $y = y_c + y_p = c_1 e^x + c_2 e^{2x} - \frac{1}{2}\sin(2x) + \frac{3}{2}\cos(2x)$.