92.445/545 Partial Differential Equations Spring 2013
Midterm Exam Solutions

Problem 1. (10 points)

Is the function given by $u(x,y) = x^2 + y^2$ a solution of the pde $yu_x - xu_y = 0$? Why or why not?

$u(x,y) = x^2 + y^2 \Rightarrow u_x = 2x$ and $u_y = 2y$.

Therefore, $yu_x - xu_y = y(2x) - x(2y) = 0$ so $\boxed{u(x,y) = x^2 + y^2 \textbf{ is } \text{a solution of the pde } yu_x - xu_y = 0.}$

Problem 2. (30 points)

Solve the equation $\dfrac{1}{x}u_x - \dfrac{1}{y}u_y = 2u$ on $x > 0, y > 0$ with the initial condition $u(x,x) = x^2$.

The initial curve Γ is given by $x = s, y = s$. On Γ we have $u(s) = s^2$. The characteristic curves satisfy the conditions $\dfrac{dx}{dt} = \dfrac{1}{x}$ and $\dfrac{dy}{dt} = -\dfrac{1}{y}$. $\dfrac{dx}{dt} = \dfrac{1}{x} \Rightarrow x\ dx = dt \Rightarrow \dfrac{x^2}{2} = t + f(s)$. Because $x = s$ on Γ (where $t = 0$), $f(s)$ must equal $\dfrac{s^2}{2}$, so $\dfrac{x^2}{2} = t + \dfrac{s^2}{2}$. Similarly, $\dfrac{y^2}{2} = -t + \dfrac{s^2}{2}$. On characteristics, $\dfrac{du}{dt} = 2u$ from the given pde. $\dfrac{du}{dt} = 2u \Rightarrow \dfrac{du}{u} = 2\ dt \Rightarrow \ln(u) = 2t + g(s) \Rightarrow u = e^{2t+g(s)} = e^{2t}\underbrace{e^{g(s)}}_{h(s)} \Rightarrow$
$u = h(s)e^{2t}$. Because $u = s^2$ on Γ (where $t = 0$), $h(s)$ must equal s^2, so $u = s^2e^{2t}$.

$\dfrac{x^2}{2} = t + \dfrac{s^2}{2}$ and $\dfrac{y^2}{2} = -t + \dfrac{s^2}{2} \Rightarrow \dfrac{x^2+y^2}{2} = s^2$ and $\dfrac{x^2-y^2}{2} = 2t$.

Therefore, $\boxed{u(x,y) = \left(\dfrac{x^2+y^2}{2}\right)e^{(x^2-y^2)/2}.}$

Problem 3. (30 points) (Pinchover and Rubinstein problem 4.9)

Solve the following Cauchy problem for the nonhomogeneous wave equation.

$$\begin{aligned} u_{tt} - u_{xx} &= 1 \qquad \text{on } -\infty < x < \infty,\ t > 0 \\ u(x,0) &= x^2 \\ u_t(x,0) &= 1 \end{aligned}$$

From D'Alembert's Formula for the nonhomogeneous wave equation, we have

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ ds + \frac{1}{2c}\int\!\!\int_{\Delta} F(\xi,\tau)\,d\xi\,d\tau$$

where Δ is the triangular region with vertices $(x-ct,0), (x+ct,0)$ and (x,t). (See equation 4.17 on page 89 of the textbook.) In this problem, $c = 1$, $f(x) = x^2$, $g(x) = 1$, $F(x,t) = 1$, and Δ is the triangular region with vertices $(x-t,0), (x+t,0)$ and (x,t). Therefore,

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\left[(x+t)^2+(x-t)^2\right]+\frac{1}{2}\int_{x-t}^{x+t}1\,ds+\frac{1}{2}\iint_{\Delta}1\,d\xi\,d\tau \\
&= x^2+t^2+\frac{1}{2}\,s|_{x-t}^{x+t}+\frac{1}{2}\,(\text{area of }\Delta) \\
&= x^2+t^2+\frac{1}{2}\left[(x+t)-(x-t)\right]+\frac{1}{2}\left(\frac{1}{2}(2t)(t)\right)
\end{aligned}
$$

so $\boxed{u(x,t)=x^2+t+\frac{3t^2}{2}}$

Problem 4. (15 points) Classify each of the following pde's as hyperbolic, elliptic, or parabolic.

Recall that the second order linear pde in 2 independent variables $au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y+fu=g$ is hyperbolic if $b^2-ac>0$, parabolic if $b^2-ac=0$, and elliptic if $b^2-ac<0$. (See page 65 of the textbook.)

(a) $u_{xx}+2u_{xy}+u_{yy}+u_x+u_y=0$

Here $a=1$, $2b=2$, and $c=1$, so $b^2-ac=1^2-1(1)=0$. Therefore, $\boxed{\text{this pde is parabolic in the entire } xy \text{ plane.}}$

(b) $u_{xx}+2u_{xy}+2u_{yy}+u_x+u_y=\sin(xy)$

Here $a=1$, $2b=2$, and $c=2$, so $b^2-ac=1^2-1(2)=-1<0$. Therefore, $\boxed{\text{this pde is elliptic in the entire } xy \text{ plane.}}$

(c) $2u_{xx}-4u_{xy}-6u_{yy}+u_x=0$

Here $a=2$, $2b=-4$, and $c=-6$, so $b^2-ac=(-2)^2-2(-6)=16>0$. Therefore, $\boxed{\text{this pde is hyperbolic in the entire } xy \text{ plane.}}$

Problem 5. (15 points) Find the canonical form of the following hyperbolic pde. Be sure to show the change of coordinates that reduces the pde to canonical form.

$$u_{xx}+6u_{xy}-16u_{yy}=0$$

As explained on page 67 of the textbook, the canonical variables ξ and η for a hyperbolic pde satisfy the equations $a\xi_x+\left(b+\sqrt{b^2-ac}\right)\xi_y=0$ and $a\eta_x+\left(b-\sqrt{b^2-ac}\right)\eta_y=0$. In this problem, these equations reduce to $\xi_x+8\xi_y=0$ and $\eta_x-2\eta_y=0$. Solving these equations by the method of characteristics, we find that $\xi=f(-8x+y)$ and $\eta=g(2x+y)$. For simplicity we take $\boxed{\xi=-8x+y}$ and $\boxed{\eta=2x+y}$. We therefore have

$$
\begin{aligned}
u_x &= w_\xi\xi_x+w_\eta\eta_x=-8w_\xi+2w_\eta \\
u_y &= w_\xi\xi_y+w_\eta\eta_y=w_\xi+w_\eta \\
u_{xx} &= -8\left[w_{\xi\xi}\xi_x+w_{\xi\eta}\eta_x\right]+2\left[w_{\eta\xi}\xi_x+w_{\eta\eta}\eta_x\right]=64w_{\xi\xi}-32w_{\xi\eta}+4w_{\eta\eta} \\
u_{xy} &= -8\left[w_{\xi\xi}\xi_y+w_{\xi\eta}\eta_y\right]+2\left[w_{\eta\xi}\xi_y+w_{\eta\eta}\eta_y\right]=-8w_{\xi\xi}-6w_{\xi\eta}+2w_{\eta\eta} \\
u_{yy} &= w_{\xi\xi}\xi_y+w_{\xi\eta}\eta_y+w_{\eta\xi}\xi_y+w_{\eta\eta}\eta_y=w_{\xi\xi}+2w_{\xi\eta}+w_{\eta\eta}
\end{aligned}
$$

Therefore, the given pde $u_{xx}+6u_{xy}-16u_{yy}=0$ becomes
$[64w_{\xi\xi}-32w_{\xi\eta}+4w_{\eta\eta}]+6\left[-8w_{\xi\xi}-6w_{\xi\eta}+2w_{\eta\eta}\right]-16\left[w_{\xi\xi}+2w_{\xi\eta}+w_{\eta\eta}\right]=0$, or $-100w_{\xi\eta}=0$, or $\boxed{w_{\xi\eta}=0}$

Extra Credit (10 points)

As mentioned in class, if u is a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ on $-\infty < x < \infty, t > 0$ for which $u \to 0, u_x \to 0$, and $u_t \to 0$ as $x \to \pm\infty$, then the energy $E = \int_{-\infty}^{\infty} \left(u_t^2 + c^2 u_x^2\right)\, dx$ is constant.

Can you find a corresponding conserved quantity $\hat{E}$ for solutions of the equation $u_{tt} - c^2 u_{xx} - bu = 0$?

Try $\hat{E} = \int_{-\infty}^{\infty} \left(u_t^2 + c^2 u_x^2 + bu^2\right)\, dx$

FOR STUDENTS ENROLLED IN 92.545.

Problem 6. (20 points)

(a) (15 points) Solve the equation $uu_x + u_y = 1$ with the initial condition $u(x, x) = 0$.
Hints: When finding the characteristic curves, solve for u before solving for x. When expressing t in terms of x and y, remember that $t = 0$ on the initial curve.

The initial curve Γ is given by $x = s, y = s$. On Γ we have $u(s) = 0$. The characteristic curves satisfy the conditions $\frac{dx}{dt} = u$ and $\frac{dy}{dt} = 1$. On characteristics, $\frac{du}{dt} = 1$ from the given pde. $\frac{du}{dt} = 1 \Rightarrow u = t + f(s)$. Because $u = 0$ on Γ (where $t = 0$), $f(s)$ must equal 0, so $u = t$. $\frac{dx}{dt} = u \Rightarrow \frac{dx}{dt} = t \Rightarrow dx = t\ dt \Rightarrow x = \frac{t^2}{2} + g(s)$. Because $x = s$ on Γ (where $t = 0$), $g(s)$ must equal s, so $x = \frac{t^2}{2} + s$. $\frac{dy}{dt} = 1 \Rightarrow y = t + h(s)$. Because $y = s$ on Γ (where $t = 0$), $h(s)$ must equal s, so $y = t + s$. $x = \frac{t^2}{2} + s$ and $y = t + s \Rightarrow x - y = \frac{t^2}{2} - t \Rightarrow t^2 - 2t + 2(y - x) = 0$
$\Rightarrow t = \frac{2 \pm \sqrt{4 - 8(y-x)}}{2} = 1 \pm \sqrt{1 - 2(y - x)}$ Because $t = 0$ on the initial curve $y = x$, we must choose $t = 1 - \sqrt{1 - 2(y - x)}$.

Therefore, $\boxed{u(x, y) = 1 - \sqrt{1 - 2(y - x)}.}$

(b) (5 points) Find the domain of the solution $u(x, y)$ you found in part a.

From the solution formula, we see that $1 - 2(y - x)$ must be nonnegative, so the domain is $\left\{(x, y) \middle|\ y \le x + \frac{1}{2}\right\}$

Problem 7. (10 points)

On Homework Assignment # 3 you showed that if u is a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ on $-\infty < x < \infty, t > 0$ for which $u \to 0, u_x \to 0$, and $u_t \to 0$ as $x \to \pm\infty$, then the energy $E = \int_{-\infty}^{\infty} \left(u_t^2 + c^2 u_x^2\right)\, dx$ is constant.

Show that if u is a solution of the equation $u_{tt} - c^2 u_{xx} + au_t = 0$ $(a > 0)$ on $-\infty < x < \infty, t > 0$ for which $u \to 0, u_x \to 0$, and $u_t \to 0$ as $x \to \pm\infty$, then $\frac{dE}{dt} \le 0$.

Note: The term au_t in the pde is a *dissipative* term that causes energy loss.

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\int_{-\infty}^{\infty}\left(u_t^2 + c^2 u_x^2\right)\ dx \\
&= \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(u_t^2 + c^2 u_x^2\right)\ dx \text{ using Leibniz's Rule} \\
&= \int_{-\infty}^{\infty}\left(2u_t u_{tt} + 2c^2 u_x u_{xt}\right)\ dx \\
&= 2\int_{-\infty}^{\infty}\left[u_t\left(c^2 u_{xx} - a u_t\right) + c^2 u_x u_{xt}\right]\ dx \text{ using the given pde} \\
&= 2c^2\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[u_t u_x\right]\ dx - 2a\int_{-\infty}^{\infty} u_t^2\ dx \\
&= 2c^2\ u_t u_x\Big|_{x\to-\infty}^{x\to\infty} - 2a\int_{-\infty}^{\infty} u_t^2\ dx \\
&= 0 - 2a\int_{-\infty}^{\infty} u_t^2\ dx \text{ because } u_x \to 0 \text{ and } u_t \to 0 \text{ as } x \to \pm\infty \\
&= -2a\int_{-\infty}^{\infty} u_t^2\ dx
\end{aligned}
$$

$u_t^2 \geq 0 \Rightarrow \int_{-\infty}^{\infty} u_t^2\ dx \geq 0$, and $a > 0$, so $\dfrac{dE}{dt} \leq 0$.