92.445/545 Partial Differential Equations Spring 2013
Homework Assignment # 1 Solutions

1. Pinchover and Rubinstein problem 1.1 Show that each of the following equations has a solution of the form $u(x, y) = f(ax + by)$ for a proper choice of constants a, b. Find the constants for each example.

$u(x, y) = f(ax + by) \Rightarrow \dfrac{\partial u}{\partial x} = f'(ax + by) \cdot \dfrac{\partial}{\partial x}[ax + by] = af'(ax + by)$ and
$\dfrac{\partial u}{\partial y} = f'(ax + by) \cdot \dfrac{\partial}{\partial y}[ax + by] = bf'(ax + by)$

(a) $u_x + 3u_y = 0$

Left side of pde: $u_x + 3u_y = af'(ax + by) + 3bf'(ax + by) = (a + 3b)f'(ax + by)$. This will equal 0 as long as $a + 3b = 0$. Thus, $f(ax + by)$ is a solution of the pde $u_x + 3u_y = 0$ for any choice of constants a and b satisfying the condition $a + 3b = 0$.

(b) $3u_x - 7u_y = 0$

Left side of pde: $3u_x - 7u_y = 3af'(ax + by) - 7bf'(ax + by) = (3a - 7b)f'(ax + by)$. This will equal 0 as long as $3a - 7b = 0$. Thus, $f(ax + by)$ is a solution of the pde $3u_x - 7u_y = 0$ for any choice of constants a and b satisfying the condition $3a - 7b = 0$.

(c) $2u_x + \pi u_y = 0$

Left side of pde: $2u_x + \pi u_y = 2af'(ax + by) + \pi bf'(ax + by) = (2a + \pi b)f'(ax + by)$. This will equal 0 as long as $2a + \pi b = 0$. Thus, $f(ax + by)$ is a solution of the pde $2u_x + \pi u_y = 0$ for any choice of constants a and b satisfying the condition $2a + \pi b = 0$.

2. Pinchover and Rubinstein problem 1.2 Show that each of the following equations has a solution of the form $u(x, y) = e^{\alpha x + \beta y}$. Find the constants α, β for each example.

$u(x, y) = e^{\alpha x + \beta y} \Rightarrow \dfrac{\partial u}{\partial x} = e^{\alpha x + \beta y} \cdot \dfrac{\partial}{\partial x}[\alpha x + \beta y] = \alpha e^{\alpha x + \beta y}$,
$\dfrac{\partial u}{\partial y} = e^{\alpha x + \beta y} \cdot \dfrac{\partial}{\partial y}[\alpha x + \beta y] = \beta e^{\alpha x + \beta y}$, $\dfrac{\partial^2 u}{\partial x^2} = \alpha e^{\alpha x + \beta y} \cdot \dfrac{\partial}{\partial x}[\alpha x + \beta y] = \alpha^2 e^{\alpha x + \beta y}$, etc.

(a) $u_x + 3u_y + u = 0$

Left side of pde: $u_x + 3u_y + u = \alpha e^{\alpha x + \beta y} + 3\beta e^{\alpha x + \beta y} + e^{\alpha x + \beta y} = (\alpha + 3\beta + 1)e^{\alpha x + \beta y}$. This will equal 0 as long as $\alpha + 3\beta + 1 = 0$. Thus, $u(x, y) = e^{\alpha x + \beta y}$ is a solution of the pde $u_x + 3u_y + u = 0$ for any choice of constants α and β satisfying the condition $\alpha + 3\beta + 1 = 0$.

(b) $u_{xx} + u_{yy} = 5e^{x-2y}$

Left side of pde: $u_{xx} + u_{yy} = \alpha^2 e^{\alpha x + \beta y} + \beta^2 e^{\alpha x + \beta y} = \left(\alpha^2 + \beta^2\right) e^{\alpha x + \beta y}$. This will equal $5e^{x-2y}$ as long as $(\alpha^2 + \beta^2)\, e^{\alpha x + \beta y} = 5e^{x-2y}$. Thus, α must equal 1 and β must equal -2. $u(x, y) = e^{x-2y}$ is a solution of the pde $u_{xx} + u_{yy} = 5e^{x-2y}$.

(c) $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0$

Left side of pde: $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = \alpha^4 e^{\alpha x + \beta y} + \beta^4 e^{\alpha x + \beta y} + 2\alpha^2\beta^2 e^{\alpha x + \beta y} = \left(\alpha^4 + \beta^4 + 2\alpha^2\beta^2\right) e^{\alpha x + \beta y}$. This will equal 0 as long as $\alpha^4 + \beta^4 + 2\alpha^2\beta^2 = 0 \Rightarrow \left(\alpha^2 + \beta^2\right)^2 = 0$. The only real values of α and β that satisfy this condition are $\alpha = \beta = 0$. Thus, $u(x, y) = e^{0x+0y} = 1$ is a solution of the pde $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0$.

3. Pinchover and Rubinstein problem 2.1 Solve the equation $u_x + u_y = 1$ with the initial condition $u(x, 0) = f(x)$.

The initial curve Γ is given by $x = s, y = 0$. On Γ we have $u(s) = f(s)$. The characteristic curves satisfy the conditions $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 1$ because the coefficients of u_x and u_y in the given pde are both equal to 1. $\frac{dx}{dt} = 1 \Rightarrow x = t + g(s)$. Because $x = s$ on Γ (where $t = 0$), $g(s)$ must equal s, so $x = t + s$. $\frac{dy}{dt} = 1 \Rightarrow y = t + h(s)$. Because $y = 0$ on Γ (where $t = 0$), $h(s)$ must equal 0, so $y = t$. On characteristics, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u_x \cdot 1 + u_y \cdot 1 = u_x + u_y = 1$ from the given pde. $\frac{du}{dt} = 1 \Rightarrow u = t + \phi(s)$. Because $u = f(s)$ on Γ (where $t = 0$), $\phi(s)$ must equal $f(s)$, so $u = t + f(s)$.
Since $y = t$ and $x = t + s$ on characteristics, we have $t = y$ and $s = x - y$. Therefore, $\boxed{u(x, y) = y + f(x - y)}$

4. Pinchover and Rubinstein problem 2.2 Solve the equation $xu_x + (x + y)u_y = 1$ with the initial condition $u(1, y) = y$. Is the solution defined everywhere?

The initial curve Γ is given by $x = 1, y = s$. On Γ we have $u(s) = s$. The characteristic curves satisfy the conditions $\frac{dx}{dt} = x$ and $\frac{dy}{dt} = x + y$. $\frac{dx}{dt} = x \Rightarrow x = g(s)e^t$. Because $x = 1$ on Γ (where $t = 0$), $g(s)$ must equal 1, so $x = e^t$. $\frac{dy}{dt} = x + y = e^t + y \Rightarrow y = e^t\,(t + h(s))$. (The ode for y is a linear first order ode. See this handout on first order linear ode's.) Because $y = s$ on Γ (where $t = 0$), $h(s)$ must equal s, so $y = e^t\,(t + s)$. On characteristics, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u_x \cdot x + u_y \cdot (x + y) = 1$ from the given pde. $\frac{du}{dt} = 1 \Rightarrow u = t + \phi(s)$. Because $u = s$ on Γ (where $t = 0$), $\phi(s)$ must equal s, so $u = t + s$.
Since $x = e^t$ and $y = e^t\,(t + s)$ on characteristics, we have $y/x = t + s$. Therefore, $\boxed{u(x, y) = y/x}$.
The domain of this solution cannot contain any points at which $x = 0$. Since the initial curve Γ is in the right half plane, this solution is only valid for $x > 0$.

5. Pinchover and Rubinstein problem 2.7
Solve the Cauchy problem equation $u_x + u_y = u^2,\ u(x, 0) = 1$.

The initial curve Γ is given by $x = s, y = 0$. On Γ we have $u(s) = 1$. The characteristic curves satisfy the conditions $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 1$ because the coefficients of u_x and u_y in the given pde are both equal to 1. $\frac{dx}{dt} = 1 \Rightarrow x = t + g(s)$. Because $x = s$ on Γ (where $t = 0$), $g(s)$ must equal s, so $x = t + s$. $\frac{dy}{dt} = 1 \Rightarrow y = t + h(s)$. Because $y = 0$ on Γ (where $t = 0$), $h(s)$ must equal 0, so $y = t$. On characteristics, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u_x \cdot 1 + u_y \cdot 1 = u_x + u_y = u^2$ from the given pde. $\frac{du}{dt} = u^2 \Rightarrow u = -\frac{1}{t + \phi(s)}$. Because $u = 1$ on Γ (where $t = 0$), $\phi(s)$ must equal -1, so $u = -\frac{1}{t-1} = \frac{1}{1-t}$.
Since $y = t$ on characteristics, we have $\boxed{u(x, y) = \frac{1}{1-y}}$

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6. Pinchover and Rubinstein problem 1.3
(a) Show that there exists a unique solution for the system $u_x = 3x^2y + y,\ u_y = x^3 + x$ together with the initial condition $u(0,0) = 0$.

$u_x = 3x^2y + y \Rightarrow u = \int \left(3x^2y + y\right)\ dx = x^3y + xy + f(y)$ (Remember that y is treated as a constant when calculating u_x.)
$u = x^3y + xy + f(y) \Rightarrow u_y = x^3 + x + f'(y)$. This means that the second equation in the given system will be satisfied if $f'(y) = 0$, or $f(y) = c$ where c is a constant. Therefore, $u(x,y) = x^3y + xy + c$.
$u(0,0) = 0 \Rightarrow 0^3(0) + 0(0) + c = 0 \Rightarrow c = 0$. Therefore,

$\boxed{u(x,y) = x^3y + xy \text{ is the only solution to the given system that satisfies the given condition } u(0,0) = 0.}$

(b) Prove that the system $u_x = 2.999999x^2y + y,\ u_y = x^3 + x$ has no solution at all.

Suppose there were a smooth function that satisfied both equations in the system.
$u_x = 2.999999x^2y + y \Rightarrow u_{xy} = 2.999999x^2 + 1$ and $u_y = x^3 + x \Rightarrow u_{yx} = 3x^2 + 1$. But u_{xy} must equal u_{yx}. Therefore, there is no solution to the given system.

7. Pinchover and Rubinstein problem 2.4 Consider the equation $yu_x - xu_y = 0 \quad (y > 0)$. Check for each of the following initial conditions whether the problem is solvable. If it is solvable, find a solution. If it is not, explain why.

In each case the initial curve Γ is given by $x = s, y = 0$. The characteristic curves satisfy the conditions $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -x$. $\frac{dx}{dt} = y \Rightarrow \frac{d^2x}{dt^2} = \frac{dy}{dt}$. But $\frac{dy}{dt} = -x$, so $\frac{d^2x}{dt^2} = -x \Rightarrow x'' + x = 0$ $\Rightarrow x = c_1\cos(t) + c_2\sin(t)$. (See this handout on constant coefficient linear ode's.)
$\frac{dx}{dt} = y \Rightarrow y = -c_1\sin(t) + c_2\cos(t)$. Because $x = s$ and $y = 0$ on Γ (where $t = 0$), c_1 must equal s and c_2 must equal 0. Therefore, the characteristic curves are given by $x = s\cos(t),\ y = -s\sin(t)$.
$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = yu_x - xu_y = 0$ from the given pde. $\frac{du}{dt} = 0 \Rightarrow u = c$ (constant) on a characteristic curve.

(a) $u(x,0) = x^2$

Because $x = s$ on Γ, this condition tells us that $u = s^2$ on Γ. Since u is constant on a characteristic curve, we have $u(t,s) = s^2$. $x = s\cos(t),\ y = -s\sin(t) \Rightarrow x^2 + y^2 = s^2\cos^2(t) + s^2\sin^2(t) = s^2$.
Therefore, $\boxed{u = x^2 + y^2}$

(b) $u(x,0) = x$

$u(x,0) = x \Rightarrow u = s$ on Γ. As shown in part (a), $x^2 + y^2 = s^2$, so $s = \sqrt{x^2+y^2}$ for x (or s) greater than 0 and $s = -\sqrt{x^2+y^2}$ for $x < 0$. Therefore, $u = \begin{cases} \sqrt{x^2+y^2} & \text{if } x > 0 \\ -\sqrt{x^2+y^2} & \text{if } x < 0 \end{cases}$

(c) $u(x,0) = x,\ x > 0$

As shown in part (b), $\boxed{u = \sqrt{x^2+y^2} \ \text{ if } x > 0}$