Finitely additive measures on o-minimal sets

Tibor Beke

University of Massachusetts

tibor_beke@uml.edu

July 27, 2009

set-up

- work topologically (over $\mathbb{R}$, the real reals)
  could also take real-closed base field
- work in the affine semi-algebraic setting
  could extend to semi-algebraic spaces, or
  o-minimal expansion of a real-closed field

$$\text{category } \mathrm{SA}_{\mathbb{R}} \begin{cases} \text{objects: affine semi-algebraic sets} \\ \text{morphisms: continuous semi-algebraic maps} \end{cases}$$

the combinatorial euler characteristic

Theorem (Hironaka; Delfs, Knebusch; Schanuel; van den Dries):

There is a unique map *eu*: $\{\text{iso classes } \mathrm{SA}_{\mathbb{R}}\} \to \mathbb{Z}$ such that

- $eu(X) = eu(Y) + eu(X - Y)$ for semi-algebraic $Y \subset X$
- $eu(X \times Y) = eu(X)\, eu(Y)$

In fact

$$eu(X) = \sum_{\alpha \in cell(X)} (-1)^{\dim \alpha}$$

in any decomposition of X into open semi-algebraic cells.

cohomological euler characteristics

Problem: Find a cohomology theory $H^* : \mathrm{SA}_{\mathbb{R}}^{op} \to K\text{-}Vect$ such that

$$eu(X) = \chi(X)$$

for all semi-algebraic X.

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim_K H^i(X)$$

what is a cohomology theory? (wishlist)

- graded contravariant functor $H^i : \mathrm{SA}_{\mathbb{R}} \to K\text{-}Vect$
  (K: field, not necessarily of characteristic zero; $i \geqslant 0$)
- Mayer-Vietoris long exact sequence

$$\ldots \to H^i(X \cup Y) \to H^i(X) \oplus H^i(Y) \to H^i(X \cap Y) \to \ldots$$

  for *certain* subspaces X, Y of $X \cup Y$

  whence $\quad \chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$

what is a cohomology theory? (wishlist, cont'd)

- Künneth theorem $H^*(X \times Y) = H^*(X) \otimes_K H^*(Y)$

$$\text{i.e.} \quad H^n(X \times Y) = \bigoplus_{i+j=n} H^i(X) \otimes_K H^j(Y)$$

whence

$$\chi(X \times Y) = \chi(X)\,\chi(Y).$$

- for a fibration
$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow \\ & & B \end{array}$$
local trivialization, Künneth and Mayer-Vietoris give

$$\chi(X) = \chi(F)\,\chi(B).$$

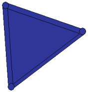
what is a cohomology theory? (wishlist, cont'd)

- $\dim_K H^i(X) = 0$ for $i > \dim(X)$
- $H^*(X)$ is a graded-commutative ring
- there are cohomology operations $H^*(X) \to H^*(X)$ (e.g. Steenrod operations, when coefficients are $\mathbb{Z}/p$)
- relative theories (cohomology of a pair, cohomology for spaces over a base; equivariant versions) etc

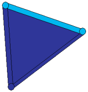
what is the question?

Find a cohomology theory $H^* : \mathrm{SA}_{\mathbb{R}}^{op} \longrightarrow K\text{-}Vect$ such that $eu(X) = \chi(X)$ for all semi-algebraic X.

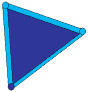
(i) compact X



(ii) locally compact X



(iii) general (not loc. compact) X



what is the answer

Find a cohomology theory $H^* : \mathrm{SA}_{\mathbb{R}}^{op} \longrightarrow K\text{-}\mathit{Vect}$ such that $eu(X) = \chi(X)$ for all semi-algebraic X.

(i) for compact X: any of simplicial, singular, sheaf, Čech, Alexander-Spanier cohomology

(ii) for locally compact X: singular cohomology with compact support; sheaf cohomology with compact support

(iii) for general (not loc. compact) X: unclear (doesn't seem to have been addressed in the literature!)

the compact case: simplicial cohomology

A ("complete", "combinatorial", "abstract") *simplicial complex* $\mathcal{S}$ is a set of subsets of a (finite) set ("vertices") such that if $X \in \mathcal{S}$ and $Y \subset X$ then $Y \in \mathcal{S}$. Has geometric realization $|\mathcal{S}|$ which is a compact polyhedron. Any compact semi-algebraic set is semi-algebraically homeomorphic to such an $|\mathcal{S}|$.

Theorem (Poincaré) $eu(|\mathcal{S}|) = \chi_{simp}(|\mathcal{S}|)$

where χ_{simp} is the cohomological euler characteristic associated to simplicial cohomology.

the compact case: simplicial cohomology

proof consider chain complex for computing $H^*_{simp}(X)$

$$\ldots \to C_{i-1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i+1} \to \ldots$$

with C_i the K-vectorspace on the set of closed i-simplices as basis

$$eu(X) = \sum_{\alpha \in \text{open cell}(X)} (-1)^{\dim \alpha} = \sum_{\alpha \in \text{closed cell}(X)} (-1)^{\dim \alpha} =$$

$$= \sum (-1)^i \dim_K C_i = \sum (-1)^i \dim_K H^i_{simp} = \chi_{simp}(X)$$

but is it a homeomorphism invariant?

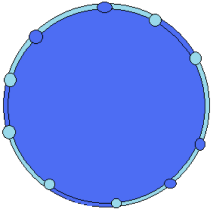
Suppose the compact polyhedra $|\mathcal{S}_1|$, $|\mathcal{S}_2|$ are homeomorphic. Are they combinatorially homeomorphic (i.e. are some subdivisions of $\mathcal{S}_1$ and $\mathcal{S}_2$ isomorphic as simplicial complexes?)

no (Milnor, 1961) The topological *Hauptvermutung* is false! Gave explicit counterexamples

yet if $|\mathcal{S}_1|$, $|\mathcal{S}_2|$ are homeomorphic then $H^*_{simp}(|\mathcal{S}_1|) = H^*_{simp}(|\mathcal{S}_2|)$ (since $H^*_{simp}(-) = H^*_{sing}(-) = H^*_{sheaf}(-)$ for finite CW-complexes)

aside Milnor's counterexample implies that two polytopes (of dimension $\geqslant 6$) can be topologically homeomorphic without being o-minimally so

what goes wrong with non-compact spaces



for X convex, *eu*(X) can take on *any* integer value while $\chi(X) = 1$ for any cohomology theory satisfying the Eilenberg-Steenrod axioms (in particular, homotopy invariance)

the locally compact case, sheaf-theoretically (1)

$H^n_c(X; A)$ is the n-th derived functor of the global sections with compact support functor Γ_c: Sh(X)→Ab (evaluated at A)

- functorial w.r.t. proper maps only (and homotopy invariant w.r.t. proper homotopy only)
- for Hausdorff, compact spaces, $H^*_c(-) = H^*(-)$
- for Hausdorff, locally compact X, open $i : U \subset X$ with closed complement $j : Z \subset X$, long exact sequence

$$\ldots \to H^n_c(U, i^*A) \to H^n_c(X, A) \to H^n_c(Z, j^*A) \to \ldots$$

the locally compact case, sheaf-theoretically (2)

- when A is the constant sheaf K, long exact sequence yields $\chi_c(X) = \chi_c(U) + \chi_c(Z)$
- applying this to the inclusion of a Hausdorff, locally compact space X in its one-point (Čech) compactification X^+

$$\chi(X^+) = \chi_c(X) + 1$$

- therefore $\chi_c\big(\ (0,1)^i\ \big) = (-1)^i$
- by induction on an open-cell decomposition,

$$eu(X) = \chi_c(X)$$

for locally compact, semi-algebraic X.

the locally compact case, simplicially (1)

• *complete simplicial complex* $\mathcal{S}$: set of subsets of a (finite) set such that if $X \in \mathcal{S}$ and $Y \subset X$ then $Y \in \mathcal{S}$.

• *locally complete simplicial complex* $\mathcal{S}$: set of subsets of a (finite) set such that if $X \subset Y \subset Z$ and $X, Z \in \mathcal{S}$ then $Y \in \mathcal{S}$. Its geometric realization $|\mathcal{S}|$ is a locally compact polyhedron, and any locally compact semi-algebraic set is semi-algebraically homeomorphic to such an $|\mathcal{S}|$.

the locally compact case, simplicially (2)

$\mathcal{S}$: locally complete simplicial complex

place a linear order $\prec$ on the vertices of $\mathcal{S}$ (to make life simple)

C_i: K-vectorspace with basis $\{X \in \mathcal{S} \mid card(X) = i+1\}$

define boundary $C_i \xrightarrow{\partial} C_{i-1}$ by $\partial = \sum_{j=0}^{i} (-1)^j \, \partial_j$
where for $k_0 \prec k_1 \prec \cdots \prec k_i$

$$\partial_j \langle k_0, k_1, \ldots, k_i \rangle =$$

$$\begin{cases} \langle k_0, k_1, \ldots, \widehat{k_j}, \ldots, k_i \rangle & \text{if } \langle k_0, k_1, \ldots, \widehat{k_j}, \ldots, k_i \rangle \in \mathcal{S} \\ 0 & \text{otherwise.} \end{cases}$$

the locally compact case, simplicially (3)

Prop $\partial \circ \partial = 0$.

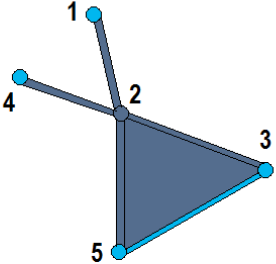
the K-linear dual of the above formulas defines a cochain complex whose homology is (by definition) $H^*_{cs}(\mathcal{S})$ ("compactly supported simplicial cohomology")

Theorem For any locally complete simplicial complex $\mathcal{S}$,

$$H^*_{cs}(\mathcal{S}) = H^*_c(|\mathcal{S}|).$$

Proof induction on cell decomposition; analogous to isomorphism of singular and cellular cohomology for CW-complexes

the locally compact case, simplicially (example)



$$\mathcal{S} = \{\langle 2,3,5\rangle, \langle 2,3\rangle, \langle 2,5\rangle, \langle 1,2\rangle, \langle 2,4\rangle, \langle 2\rangle\}$$

$$\partial\langle 2,3,5\rangle = -\langle 2,5\rangle + \langle 2,3\rangle$$

$$\partial\langle 2,5\rangle = -\langle 2\rangle$$

$$\partial\langle 2,3\rangle = -\langle 2\rangle$$

$$\partial\langle 1,2\rangle = \langle 2\rangle$$

$$\partial\langle 2,4\rangle = -\langle 2\rangle$$

$\langle 1,2\rangle + \langle 2,4\rangle$ is a cycle representing one of the generators of H^1_{cs}

$eu = -2 = \chi_{cs}$ realized combinatorially!

three drawbacks of simplicial complexes (1)

No canonical boundary (save with mod 2 coefficients)!

$$\star \longrightarrow \bullet$$

$\partial(\{\star, \bullet\}) = \{\star\} - \{\bullet\}$?

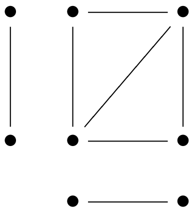
or

$\partial(\{\star, \bullet\}) = \{\bullet\} - \{\star\}$?

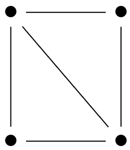
three drawbacks of simplicial complexes (2)

A morphism of simplicial complexes $\mathcal{S}_1 \to \mathcal{S}_2$ is a map of vertices that takes a distinguished subset into a distinguished subset (not necessarily injectively).

Categorical products don't exist!

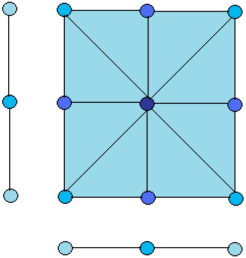


or



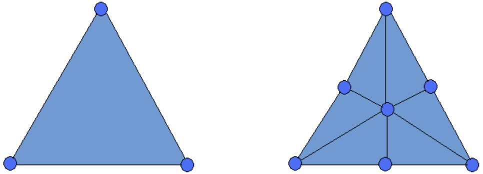
?

one barycentric subdivision solves some of these



three drawbacks of simplicial complexes (3)

but no morphism between a complex and its subdivision (underlying their topological isomorphism)



aside: simplicial sets

simplicial set: functor $\Delta^{op} \to$Set

$$\Delta = \text{category} \begin{cases} \text{objects} & \{0, 1, \ldots, n\} \quad n \in \mathbb{N} \\ \text{morphisms} & \text{non-decreasing maps} \end{cases}$$

functor $\{$ simplicial complexes $\} \xrightarrow{nerve} \{$ simplicial sets $\}$

(nerve of the poset of faces; also describable via the local ordering induced on the vertices of the first subdivision of the simplicial complex)

where we are

space	compact $\mathrm{SA}_{\mathbb{R}}$	loc. compact $\mathrm{SA}_{\mathbb{R}}$
triangulated model	complete simplicial complex	locally complete simplicial complex
discrete model	simplicial set	incomplete simplicial set (?)
cohomology theory (topological)	sheaf cohomology	sheaf cohomology with compact support
cohomology theory (combinatorial)	simplicial cohomology	compactly supported simplicial cohomology

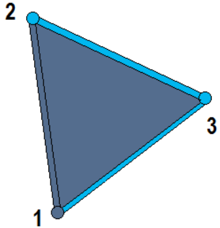
general semi-algebraic sets (1)

• *complete simplicial complex* $\mathcal{S}$: set of subsets of a (finite) set such that if $X \in \mathcal{S}$ and $Y \subset X$ then $Y \in \mathcal{S}$.

• *locally complete simplicial complex* $\mathcal{S}$: set of subsets of a (finite) set such that if $X \subset Y \subset Z$ and $X, Z \in \mathcal{S}$ then $Y \in \mathcal{S}$.

• *open-cell complex* $\mathcal{S}$: set of subsets of a (finite) set. Its geometric realization $|\mathcal{S}|$ is a piecewise linear semi-algebraic set, and any semi-algebraic set is semi-algebraically homeomorphic to such an $|\mathcal{S}|$.

general semi-algebraic sets (2)



$$\{\langle 1,2,3\rangle, \langle 1,2\rangle, \langle 1\rangle\}$$

no evident boundary operator ∂ on cells such that $\partial \circ \partial = 0$

general semi-algebraic sets (confession)

- *possibly* one has $eu(X) = \chi_c(X)$ for all semi-algebraic X (whether locally compact or not)

hard to determine $H_c^*(X)$ for a topological space that's not locally compact!

(compact subsets do not then form a "paracompactifying family of supports"; some basic sheaf-theoretic tricks break down, e.g. soft sheaves need not be acyclic etc)

general semi-algebraic sets (3)

Prop Any semi-algebraic space X has a canonical decomposition (as set) $X = X_0 \sqcup X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$ such that

- the X_k are semi-algebraic and locally compact
- X_k is open and dense in $\sqcup_{i \geqslant k} X_i$
- $\dim X_k - \dim X_{k+1} \geqslant 2$
- if X is semi-algebraically triangulated, the decomposition respects the triangulation.

proof iterate

$$X_0 := \left\{ x \in X \mid \text{for some } \epsilon > 0, \quad X \cap \overline{D}(x, \epsilon) \text{ is compact} \right\}$$

general semi-algebraic sets (4)

definition $H^*_{sa}(X) := \bigoplus_{i \in \text{strata}} H^*_c(X_i)$

- $eu(X) = \chi_{sa}(X)$ for all semi-algebraic X
- $H^*_{sa}(X) = H^*_c(X)$ for X locally compact
- $H^*_{sa}(-)$ has main properties of compactly supported sheaf cohomology
- is describable combinatorially if X is semi-algebraically triangulated
- but functoriality is restricted.

Schanuel's category $\mathcal{S}$

objects	$X \subseteq \mathbb{R}^n$ defined by a boolean combination of linear inequalities
morphisms	functions with definable graph

Theorem (Schanuel) $K_0(\mathcal{S}) = \mathbb{Z} \oplus \mathbb{Z}$

The Grothendieck semiring is generated by $(0, 1)$ and $(0, \infty)$.

Theorem There exist two cohomology theories $H_0^*(-)$, $H_\infty^*(-)$ on Schanuel's category such that $eu(X) = \langle H_0^*(X); H_\infty^*(X)\rangle$ for every $X \in \mathcal{S}$.

Under the inclusion $i : \mathcal{S} \subset \mathrm{SA}_{\mathbb{R}}$, $H_{sa}^*(iX) = H_0^*(X) \oplus H_\infty^*(X)$

finitely additive measures on o-minimal sets

Fix an o-minimal expansion of $\langle \mathbb{R}, +, \times, >, 1, 0 \rangle$ and let $\mathcal{B}$ be the collection of bounded definable sets. A (real-valued) finitely additive measure is a map $\mu : \mathcal{B} \to \mathbb{R}$ satisfying

- $\mu(X \sqcup Y) = \mu(X) + \mu(Y)$
- $\mu(X) = \mu(Y)$ if X, Y are isometric.

Goal: investigate a set of particularly nice finitely additive measures, the *quermassintegrals* or *Hadwiger measures* μ_i ($i \in \mathbb{N}$).

Hadwiger measures: examples

closed unit disk

$\mu_0 = 1$

$\mu_1 = 2\pi$

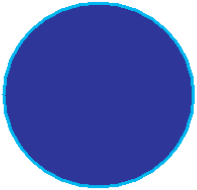
$\mu_2 = \pi$

$\mu_i = 0$ for $i > 2$

NB: this is the high school normalization of μ_1

Hadwiger measures: examples (cont'd)

open unit disk



$$\mu_0 = 1$$

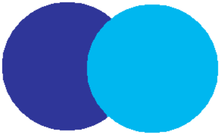
$$\mu_1 = -2\pi$$

$$\mu_2 = \pi$$

$$\mu_i = 0 \text{ for } i > 2$$

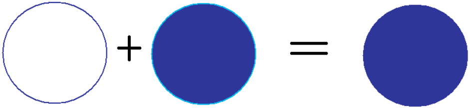
NB: this is the high school normalization of μ_1

Hadwiger measures: examples (cont'd)



additivity of "perimeter" (closed lune + open unit disk)

Hadwiger measures: examples (cont'd)



$$? + (-2\pi) = 2\pi$$

arc length is twice the usual one?

Hadwiger's formula (1)

Let $X \subset \mathbb{R}^n$ be a bounded definable set and $0 \leqslant k \leqslant n$.

$$\mu_k(X) := \int_{H \in AffGr(n,n-k)} eu(X \cap H)\; d\nu_{n,n-k}$$

where $AffGr(n, n-k)$ is the affine Grassmannian (the space of affine subspaces of dimension $n-k$ in $\mathbb{R}^n$) and $\nu_{n,n-k}$ is a suitable $E(n)$-invariant measure on it. ($E(n)$ is the group of euclidean motions of $\mathbb{R}^n$.)

For $X \subset \mathbb{R}^n$, set $\mu_k(X) = 0$ for $k > n$.

Hadwiger's formula (2)

$$\mu_k(X) := \int_{H \in AffGr(n,n-k)} eu(X \cap H)\, d\nu_{n,n-k}$$

- for each $i \in \mathbb{Z}$, the locus $\{H \in AffGr(n, n-k) \mid eu(X \cap H) = i\}$ is definable (hence $\nu_{n,n-k}$-measurable)
- finite additivity of μ_k follows from finite additivity of *eu*
- rigid motion invariance of μ_k for $X \subset \mathbb{R}^n$ follows from rigid motion invariance of $\nu_{n,n-k}$
- Careful normalization of the $\nu_{n,n-k}$ is needed to ensure that μ_k is independent of the ambient dimension n.

Hadwiger measures: basic properties

- $\mu_0(X) = eu(X)$
- $\mu_k(X)$ is very non-trivial for $0 < k < \dim(X)$
- $\mu_k(X)$ is the k-dim Lebesgue-Minkowski content when $k = \dim(X)$
- $\mu_k(X) = 0$ for $k > \dim(X)$.
- Scales as $\mu_k(\lambda X) = \lambda^k \mu_k(X)$ for X of all dimensions.

Hadwiger measures: history

- Steiner: intrinsic volumes of convex bodies
- Minkowski: mixed volumes
- analytic formulas for the euler characteristic and other intrinsic volumes (Gauss–Bonnet: smooth surfaces in $\mathbb{R}^3$; Chern: Riemannian manifolds; Federer: generalized curvature formulas; Fu: extension to certain spaces with singularities etc; Alesker: algebras of valuations)
- Hadwiger: Hadwiger's formula; axiomatic characterization of Hadwiger measures on convex bodies

on the positive side (e.g. valuations on the lattice of finite unions of compact, convex subsets of euclidean space) no need to stay within o-minimal structures

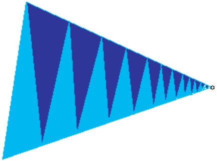
Hadwiger measures: countable additivity fails

There are counterexamples even within Schanuel's category.
For $\mu_0 = eu$:

$$(-\infty, +\infty) = \dots(-2,-1)[-1](-1,0)[0](0,1)[1](1,2)\dots$$

$$-1 = \dots - 1 + 1 - 1 + 1 - 1 \dots$$

To get a counterexample for e.g. μ_1, embed an unrectifiable boundary inside a definable set:



closed

open

Hadwiger measures: variation with parameters

Let $X \xrightarrow{f} Y$ be definable.

Then $Y \to \mathbb{Z}$ taking $y \mapsto eu\big(f^{-1}(y)\big)$ is definable.

guess: for $k > 0$, in no o-minimal expansion of the reals do the μ_k vary definably in all definable families.

Hadwiger measures: to do (1)

• Is there a "natural" boolean algebra of subsets of the $\mathbb{R}^n$ (closed under products but not necessarily projections) to which the Hadwiger measures on various o-minimal geometries can be consistently extended?

"natural" $:=$ the condition for a set to be Hadwiger measurable should be formulable in terms of (local or global) geometric properties of the set (as opposed to its generating an o-minimal expansion etc)

Hadwiger measures: to do (2)

- **define** $K(def_n)$ as the free abelian group on $\{[X] \mid \text{ bounded definable } X \subset \mathbb{R}^n\}$ modulo $[X] = [Y] + [X - Y]$ for $X \subset Y$ and $[X] = [Y]$ for X, Y isometric
- **define** the *scissors* group $Sci(def_n)$ as $K(def_n)$ modulo $[Y] = 0$ for $\dim(Y) < n$
- How big are $K(def_n)$ and $Sci(def_n)$? Is the natural map $Sci(polytopes_n) \to Sci(def_n)$ injective?
- research originating in Hilbert's 3rd problem resulted in a lot of information about scissors congruence groups of polytopes in various geometries, for example:
  the kernel of $Sci(polytopes_3) \xrightarrow{vol} \mathbb{R}$ is uncountable (detected by Dehn invariants)

Hadwiger measures: one measure to rule them all

- Prove
$$\mu_n(X \times Y) = \sum_{i+j=n} \mu_i(X)\,\mu_j(Y) \qquad (*)$$
for all definable X, Y. (Known for convex, compact X, Y!)
- Set $\mu(X) = \sum_{k=0}^{\infty} \mu_k(X)$. Note $\mu(X) < \infty$.

  $(*)$ is same as saying that $\mu(X \times Y) = \mu(X)\,\mu(Y)$.
- For any given X, $\mu_i(X)$ can be reconstructed from the degree $\dim(X)$ polynomial
$$\mu(\lambda X) = \sum_{k=0}^{\infty} \lambda^k \mu_k(X)$$
and its derivatives.

Hadwiger measures: to do (3)

- **theorem:** *eu* is the unique $\mathbb{Z}$-valued finitely additive and multiplicative homeomorphism invariant of o-minimal sets.
- **(should be true)** the set of $\mathbb{R}$-valued finitely additive and multiplicative isometry invariant [continuous] measures on o-minimal sets is parametrized by one real parameter.

  this corresponds to the choice of a scaling parameter c (equivalently, size of $[0, 1]$); on convex bodies, the measures $\sum_{k=0}^{\infty} c^k \mu_k$ are the only finitely additive-multiplicative ones that are continuous w.r.t. Minkowski metric
- what "should be true" is quite likely to be false without a continuity assumption; not clear what form that should take
- but enough to do it for **definable** convergence of definable sets — much easier!