# SIMPLICIAL TORSORS

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Abstract. The interpretation by Duskin and Glenn of abelian sheaf cohomology as connected components of a category of torsors is extended to homotopy classes. This is simultaneously an extension of Verdier's version of Čech cohomology to homotopy.

## Introduction

Let $\mathcal{E}$ be a Grothendieck topos, $A$ an abelian group object in $\mathcal{E}$. Duskin [4], completed by Glenn [8], provide a combinatorial interpretation of $H^n(\mathcal{E}, A)$, the $n^{\text{th}}$ cohomology group of $\mathcal{E}$ with coefficients in $A$, as the set of connected components of a suitable category of simplicial objects over $K(A, n)$, the canonical Eilenberg–MacLane "space" (that is, simplicial object in $\mathcal{E}$) of degree $n$ corresponding to $A$. When $n = 1$, it is not necessary that $A$ be abelian, and the formalism reduces to the well-known case of torsors, i.e. (isomorphism classes of) principal homogeneous $A$-spaces. The proof of Duskin and Glenn proceeds by showing that $\pi_0\mathsf{tors}(K(A, n))$ is a universal $\delta$-functor, and depends both on the additive nature of the situation and the delicate — and beautiful — geometry of torsors over the canonical model of an Eilenberg–MacLane space.

This paper was inspired by the question: what do the (connected components of the) category of torsors enumerate over a base that is not necessarily an Eilenberg–MacLane space? The answer is surprisingly easy once the problem is reduced to formal homotopy theory. Part of this ease is slight of hand, however. First of all, one needs to borrow results from the homotopy theory of simplicial sheaves; and secondly, we dodge the combinatorial elaborations (and economies) on the notion of simplicial torsor over $X$ that become possible only when the simplicial geometry of $X$ possesses distinguished features. These elaborations, including the notion of $K(A, n)$-torsor due to Duskin, will be mentioned only briefly in section 2.

All four ingredients of the proof of our main Thm. 1 below could have been written down by Grothendieck's school in the 60's and 70's, and indeed they *nearly* were. Let us list these ingredients right away.

**Proposition 1.** Let $\mathcal{M}$ be a category with a distinguished subcategory $\mathsf{W}$, which we assume to contain all the identity morphisms of $\mathcal{M}$. Write $\mathsf{ho}_{\mathcal{M}}$ for $\mathcal{M}[\mathsf{W}^{-1}]$, the category

1991 *Mathematics Subject Classification.* 18G30, 55U35.

The author was supported by an NWO Fellowship at Utrecht University, the Netherlands, during the preparation of this work.

obtained from $\mathcal{M}$ by formally inverting the morphisms in $\mathsf{W}$. For any objects $U$, $X$ of $\mathcal{M}$, let $\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{M}}(U, X)$ be the category whose objects are diagrams ("right fractions") in $\mathcal{M}$ of the form $U \xleftarrow{w} C \xrightarrow{f} X$ with $w \in \mathsf{W}$ but $C$, $f$ being arbitrary, while a morphism from $U \leftarrow C_1 \rightarrow X$ to $U \leftarrow C_2 \rightarrow X$ is a map $C_1 \rightarrow C_2$ that makes

$$\begin{tikzcd}
& C_1 \arrow[dl] \arrow[dr] \arrow[dd] & \\
U & & X \\
& C_2 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commute. Write $\pi_0\mathcal{C}$ for the class of connected components of a category $\mathcal{C}$ (equivalently, the connected components of the nerve of $\mathcal{C}$). There is a canonical map $\pi_0\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{M}}(U, X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U, X)$. This map is a bijection if the following condition is satisfied: for any fraction $\mathcal{F} := P \xrightarrow{f} B \xleftarrow{w} Q$, $w \in \mathsf{W}$, the corresponding category of completions, $\mathsf{comp}_{\mathcal{F}}$, is non-empty and connected. Here $\mathsf{comp}_{\mathcal{F}}$ has as objects all commutative diagrams

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "u"'] & Q \arrow[d, "w"] \\
P \arrow[r, "f"'] & B
\end{tikzcd}$$

with $u \in \mathsf{W}$, and a morphism between two such is an arrow $C_1 \xrightarrow{h} C_2$ making

$$\begin{tikzcd}
C_1 \arrow[dr, "h"] \arrow[drr] \arrow[ddr, "u_1"'] & & \\
& C_2 \arrow[r] \arrow[d, "u_2"] & Q \arrow[d, "w"] \\
& P \arrow[r, "f"'] & B
\end{tikzcd}$$

commute (note $u_1, u_2, w \in \mathsf{W}$). This condition is satisfied if $\mathsf{W}$ is closed under pullback in $\mathcal{M}$.

This variant on the classical calculus of fractions, together with K. Brown's [3] local homotopy theory of simplicial sheaves, is used to prove

**Proposition 2.** Let $\mathcal{E}$ be a Grothendieck topos, and let $\mathsf{W}$ be the class of maps in $\mathcal{E}^{\Delta^{\mathrm{op}}}$ that are weak equivalences in the local sense, i.e. that induce isomorphisms on the sheaves of homotopy groups with arbitrary (local) basepoints. Denote by LAFib the subcategory of $\mathsf{W}$ consisting of weak equivalences that are also Kan fibrations in the local, or "internal" sense. (Morphisms in LAFib are usually called *acyclic* or *trivial* local fibrations.) Then the canonical $\pi_0\mathsf{frac}_{\mathrm{LAFib}^{-1}\mathcal{E}^{\Delta^{\mathrm{op}}}}(U, X) \xrightarrow{\iota} \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(U, X)$ is a bijection provided $U$ and $X$ are locally fibrant, i.e. satisfy Kan's simplicial extension condition in the local sense.

The next proposition brings in the term "simplicial torsor". It holds for a number of variants on the definition (which will all be duly recalled in the course of the proofs). Some of these variants make the proofs easy, others are useful for describing algebraic manipulations on torsors, and some permit finitary representations of cohomology classes — and more generally, of certain types of homotopy classes. Under any definition, one will have

**Proposition 3.** Write $\mathsf{ST}(X)$ for the category of simplicial torsors over $X$, and $\mathbf{1}$ for the terminal object of $\mathcal{E}^{\Delta^{\mathrm{op}}}$. There exists a natural commutative diagram (of sets and mappings)

$$\begin{tikzcd} \pi_0\mathsf{frac}_{\mathrm{LAFib}^{-1}\,\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X) \arrow[rr, two heads, "\phi"] \arrow[dr, "\iota"'] & & \pi_0\mathsf{ST}(X) \arrow[dl] \\ & \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X) & \end{tikzcd}$$

and $\phi$ is a surjection.

One has instantaneously

**Theorem 1.** Let $X$ be a locally fibrant simplicial object in the Grothendieck topos $\mathcal{E}$. Then there is a canonical bijection

$$\pi_0\mathsf{ST}(X) \to \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$$

Indeed, all arrows in Proposition 3 are bijections by Proposition 2. □

To recover the relation to abelian cohomology, recall

**Proposition 4.** Let $A$ be an abelian group object in $\mathcal{E}$, $n \in \mathbb{N}$, and write $K(A, n)$ for a simplicial Eilenberg–MacLane object of degree $n$ corresponding to $A$. Then there is a canonical isomorphism $\mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, K(A, n)) \to H^n(\mathcal{E}, A)$. When $n = 1$, $A$ need not be abelian, and the isomorphism still holds between non-abelian cohomology in degree 1 and homotopy classes in $\mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}$.

*Proof.* Without loss of generality, we may assume $K(A, n)$ to be the canonical model of an Eilenberg–MacLane space: the simplicial abelian object that is the "denormalization" of the chain complex consisting of $A$ alone in degree $n$. Then one has a chain of adjunctions ($n \geqslant 0$, $\mathcal{D}$ being the derived category of abelian objects in $\mathcal{E}$)

$$R^n\Gamma(A) \cong \mathrm{Ext}^n_{\mathrm{Ab}(\mathcal{E})}(\mathbb{Z}, A) \cong \mathcal{D}(\mathbb{Z}, A[n]) \cong \mathsf{ho}_{\mathrm{Ab}(\mathcal{E})^{\Delta^{\mathrm{op}}}}(\mathbb{Z}, K(A, n)) \cong \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, K(A, n))$$

between the Quillen model categories of $\mathbb{N}$-indexed chain complexes of abelian group objects, resp. simplicial abelian group objects resp. simplicial objects in the topos $\mathcal{E}$. See Jardine [9] for more details, and Jardine [11] for the statement on non-abelian $H^1$. □

The homotopical approach of the present paper is quite robust, allowing one to experiment with combinatorial expressions for other global homotopy sets (or groups). The argument has a "modular" structure: each of the last three propositions enjoys its own natural level of generality in abstract homotopy, not being limited to simplicial objects. For example, the reason for breaking Prop. 3 off of Prop. 2 in the above manner is that Prop. 2 is a local argument, while Prop. 3 needs a bit of set theory. It is essentially equivalent to constructing enough fibrant objects in a Quillen model category.

The next section contains the proofs; they are brief, given some background in simplicial and/or axiomatic homotopy theory. (Prehistory and elaborations are also provided.) We then digress on variants such as hypercovers and Čech cohomology, but full discussion of these requires a separate paper. The last section is concerned purely with calculi of fractions and their role in computing homotopy classes.

## 1. Proofs

**Proof of Proposition 1.** Let us denote by $\mathsf{ho}$ the localization functor $\mathcal{M} \xrightarrow{\mathsf{ho}} \mathsf{ho}_{\mathcal{M}} := \mathcal{M}[\mathsf{W}^{-1}]$. This functor induces the map (of classes, in general[1]) $\pi_0\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{M}}(U,X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U,X)$ by sending a representative $U \xleftarrow{w} C \xrightarrow{f} X$ of a connected component to $U \xrightarrow{(\mathsf{ho}w)^{-1}} C \xrightarrow{\mathsf{ho}f} X$; this is well-defined, since if the left-hand diagram commutes, so does the one on the right:

$$\begin{tikzcd}[column sep=small]
 & C_1 \arrow[dl, "w_1"'] \arrow[dd, "c"] \arrow[dr, "f_1"] & & & & C_1 \arrow[dd, "\mathsf{ho}c"'] \arrow[dr, "\mathsf{ho}f_1"] & \\
U & & X & \overset{\gamma}{\Longmapsto} & U \arrow[ur, "(\mathsf{ho}w_1)^{-1}"] \arrow[dr, "(\mathsf{ho}w_2)^{-1}"'] & & X \\
 & C_2 \arrow[ul, "w_2"] \arrow[ur, "f_2"'] & & & & C_2 \arrow[ur, "\mathsf{ho}f_2"'] &
\end{tikzcd}$$

To prove that $\pi_0\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{M}}(U,X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U,X)$ is a bijection when $\mathcal{M},\mathsf{W}$ satisfy the condition of the proposition, i.e. the category of completions of every left fraction is non-empty and connected, we will explicitly define a category $\mathsf{W}^{-1}\mathcal{M}$ with connected components of right fractions as hom-sets, together with a functor $\mathcal{M} \to \mathsf{W}^{-1}\mathcal{M}$, and prove it has the universal property required of $\mathsf{ho}_{\mathcal{M}}$. Let $\mathsf{W}^{-1}\mathcal{M}$ have the same objects as $\mathcal{M}$, and let the hom-set $\mathsf{W}^{-1}\mathcal{M}(U,X)$ be $\pi_0\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{M}}(U,X)$. To compose morphisms $U \rightsquigarrow X$ and $X \rightsquigarrow Y$, select representatives of the components and fill in the inner diamond below (possible by the assumption that $\mathsf{comp}$ is non-empty):

$$\begin{tikzcd}[column sep=small]
 & & C \arrow[dl, dotted, "\sim"'] \arrow[dr, dotted] & & \\
 & C_1 \arrow[dl, "\sim"'] \arrow[dr] & & C_2 \arrow[dl, "\sim"'] \arrow[dr] & \\
U & & X & & Y
\end{tikzcd}$$

where the tilde indicates morphisms belonging to $\mathsf{W}$. The composition is the connected component of the leg-wise composed $U \overset{\sim}{\longleftarrow} C \to Y$. This obviously depends only on the component of $C$ in $\mathsf{comp}_{C_1 \to X \overset{\sim}{\longleftarrow} C_2}$, which is unique, and an easy verification shows that it only depends on the connected component of the representatives $U \overset{\sim}{\longleftarrow} C_1 \to X$,

[1]We work freely with classes, including categories that are not locally small, i.e. may possess a proper class of arrows between two objects, since both localized categories and categories of fractions may be such. The desired conclusion is an equivalence between two (locally not necessarily small) categories.

$X \xleftarrow{\sim} C_2 \to Y$. Associativity, units[2] and the universal property follow as for the Gabriel–Zisman calculus.

To verify the last sentence of the claim, note that if $\mathsf{W}$ is closed in $\mathcal{M}$ under pullback, then each of the categories $\mathsf{comp}_{\mathcal{F}}$, where $\mathcal{F}$ is some left fraction $P \xrightarrow{f} B \xleftarrow{w} Q$, has a terminal object; a fortiori it is non-empty and connected. □

*Remark* 1.1. See section 3 for a discussion of other calculi of fractions, including a beautiful two-sided variant and its homotopical extension, both due to Dwyer and Kan.

**Proof of Proposition 2.** Denote by $\mathcal{E}_F^{\Delta^{\mathrm{op}}}$ the full subcategory of $\mathcal{E}^{\Delta^{\mathrm{op}}}$ whose objects are locally fibrant, and write $\mathrm{LAFib}_F$ for the class $\mathrm{LAFib} \cap \mathcal{E}_F^{\Delta^{\mathrm{op}}}$ ("local acyclic fibrations between locally fibrant objects"). The inclusion $(\mathcal{E}_F^{\Delta^{\mathrm{op}}}, \mathrm{LAFib}_F) \hookrightarrow (\mathcal{E}^{\Delta^{\mathrm{op}}}, \mathsf{W})$ induces a functor $\mathcal{E}_F^{\Delta^{\mathrm{op}}}[\mathrm{LAFib}_F^{-1}] \to \mathcal{E}^{\Delta^{\mathrm{op}}}[\mathsf{W}^{-1}] =: \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}$, and a theorem of K. Brown [3] asserts that this is an equivalence of categories.

Acyclic local fibrations are preserved by pullbacks. This is classical for $Set^{\Delta^{\mathrm{op}}}$; it follows e.g. from the fact that acyclic fibrations of simplicial sets can be defined by a lifting condition. But that implies the same for $\mathcal{E}^{\Delta^{\mathrm{op}}}$, $\mathcal{E}$ being any cocomplete topos, since the property of being a local acyclic fibration is geometric — it can be defined by (countably many) geometric sentences (of countable length, in fact). $\mathrm{LAFib}_F$ is also preserved by pullbacks in $\mathcal{E}_F^{\Delta^{\mathrm{op}}}$, since the total space of a local fibration is locally fibrant if the base is so. From Prop. 1 it follows that the canonical

$$\pi_0 \mathsf{frac}_{\mathrm{LAFib}_F^{-1}\, \mathcal{E}_F^{\Delta^{\mathrm{op}}}}(U, X) \to \mathcal{E}_F^{\Delta^{\mathrm{op}}}[\mathrm{LAFib}_F^{-1}](U, X)$$

is a bijection for arbitrary locally fibrant $U$, $X$. But for such $U$, $X$, $\mathsf{frac}_{\mathrm{LAFib}_F^{-1}\, \mathcal{E}_F^{\Delta^{\mathrm{op}}}}(U, X)$ is of course the same as $\mathsf{frac}_{\mathrm{LAFib}^{-1}\, \mathcal{E}^{\Delta^{\mathrm{op}}}}(U, X)$.

*Remark* 1.2. Ken Brown established his theorem in his axiomatic setting of a "category of fibrant objects"; see also Jardine [9]. Curiously, this is the only place in the argument where a concept of homotopy is needed other than the one hiding in $\pi_0$ of a category: either simplicial homotopy of simplicial sheaves or the Quillen–Brown notion of "right homotopy" via path objects.

**Proof of Proposition 3.** We begin with one of several approaches to simplicial torsors.

**Definition 1.3.** Let $X$ be any simplicial object in $\mathcal{E}$. A *simplicial torsor over* $X$ is a local fibration $C \xrightarrow{f} X$ where $C$ is "of the weak homotopy type of the point", meaning that the canonical $C \to \mathbf{1}$ belongs to $\mathsf{W}$. Maps of torsors are simplicial maps over $X$, i.e. $\mathsf{ST}(X)$ is a full subcategory of $\mathcal{E}^{\Delta^{\mathrm{op}}}/X$.

[2] It is here that we use the assumption that $\mathsf{W}$ contains all the identity morphisms of $\mathcal{M}$.

Thus $\mathsf{ST}(X)$ can be thought of as a full subcategory of $\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$, and in the diagram

$$\begin{tikzcd}
\pi_0\mathsf{frac}_{\mathrm{LAFib}^{-1}\,\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X) \arrow[rr, two heads, "\phi"] \arrow[dr, "\iota"'] & & \pi_0\mathsf{ST}(X) \arrow[dl, "\gamma"] \\
& \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X) &
\end{tikzcd}$$

$\gamma$ is just the restriction of $\pi_0\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{M}}(\mathbf{1}, X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(\mathbf{1}, X)$ defined as in Prop. 1: $\mathbf{1} \xleftarrow{w} C \xrightarrow{f} X$ is sent to $\mathbf{1} \xrightarrow{(\mathsf{ho}w)^{-1}} C \xrightarrow{\mathsf{ho}f} X$. We still have to define $\phi$.

**Historical background.** Return for a moment to *SSet*, the category of simplicial sets. The class $\mathcal{F}$ of Kan fibrations and $\mathcal{A}$ of acyclic cofibrations (i.e. monomorphisms that are also weak equivalences, the "anodyne extensions" of Gabriel–Zisman) form a weak factorization system in the following sense: in any commutative square

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, tail, "a"'] & \bullet \arrow[d, two heads, "f"] \\
\bullet \arrow[r] \arrow[ur, dotted] & \bullet
\end{tikzcd}$$

with $a \in \mathcal{A}$ and $f \in \mathcal{F}$, a diagonal lift exists that makes both triangles commute (we say that "$f$ has the right lifting property against $a$" or "$a$ has the left lifting property against $f$"); moreover, $\mathcal{A}$ is exactly the class of maps having the left lifting property w.r.t. every element of $\mathcal{F}$, and dually for $\mathcal{F}$ against $\mathcal{A}$; and finally any simplicial map can be factored (not uniquely) as an acyclic cofibration followed by a Kan fibration.

These facts form part of Quillen's axioms for a "closed homotopy model category". (The full set of Quillen's axioms prescribes two weak factorization systems, $\langle$acyclic cofibrations, fibrations$\rangle$ and $\langle$cofibrations, acyclic fibrations$\rangle$, interacting in suitable ways.) Note that both acyclic monomorphisms and Kan fibrations can be defined by sentences of geometric logic. When interpreted in the intrinsic (i.e. local) sense in the topos $\mathcal{E}^{\Delta^{\mathrm{op}}}$, they single out the following two classes of maps: those monomorphisms that are also local weak equivalences (let us continue to call them *acyclic monos*) resp. the local fibrations. It is no longer the case that acyclic monos and local fibrations form a weak factorization system. However, it was the great insight of Joyal [13] that the class of maps that do have the right lifting property against all acyclic monos will make up a weak factorization system with them — and in fact will satisfy the rest of Quillen's axioms. This class, characterized now solely by a lifting property, turns out to be a subclass of local fibrations. Its members are often called *global fibrations.*

Seemingly the only way to prove these claims is to reduce their local parts to the case of $\mathcal{E} = Set$, and use somewhat involved transfinite combinatorics for the global ones. Joyal's original proof, dating from the early 80's [13], was never published, but see Jardine [10], [12] for careful expositions. Beke [2] shows just how automatic the reduction to *Set* can be made for other homotopy model categories too. All that is needed for our present purposes, however, is an easy

**Lemma 1.4.** *For any Grothendieck topos $\mathcal{E}$, there exists a set $A$ of acyclic monos in $\mathcal{E}^{\Delta^{op}}$ such that any map having the right lifting property against all members of $A$ must be a local fibration.*

*Proof.* Choose a small site $(\mathcal{C}, J)$ of definition for $\mathcal{E}$, and write $\mathcal{O}$ for the set of objects of $\mathcal{C}$ (thought of as a discrete category). Consider the sequence of adjunctions

$$\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J) \underset{i}{\overset{\ell}{\leftrightarrows}} \mathrm{Pre}(\mathcal{C}) \underset{r}{\overset{L_K}{\leftrightarrows}} \mathrm{Pre}(\mathcal{O})$$

where the first one is the inclusion of sheaves into presheaves, and the second one is the restriction functor (induced by the inclusion $\mathcal{O} \hookrightarrow \mathcal{C}$) and its left adjoint $L_K$, which is a left Kan extension. Let $I$ be the set of morphisms in $\mathrm{Pre}(\mathcal{O})^{\Delta^{\mathrm{op}}} = SSet^{\mathcal{O}}$ that belong, at every object of $\mathcal{O}$, to some fixed generating set of acyclic cofibrations for $SSet$ (e.g. the inclusion of $(n,k)$-horns in the $n$-simplex, $0 \leqslant k \leqslant n+1$, all $0 < n$). Set $A = \ell(L_K(I))$, the image of this set under the composite $\ell \circ L_K$. From the explicit description of $L_K$, one sees that $L_K(I)$ is made up of local acyclic monos, and these are preserved by the inverse image part of any geometric morphism, in particular, by sheafification $\ell$.

If $f \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ has the right lifting property against all members of $A$ then, by adjunction, $r(i(f))$ has the right lifting property against all members of $I$ in $\mathrm{Pre}(\mathcal{O})^{\Delta^{\mathrm{op}}} = SSet^{\mathcal{O}}$. By the way $I$ was chosen, that amounts to $r(i(f))$ being a Kan fibration in $SSet$ at every object of $\mathcal{O}$. By the definition of local properties in a presheaf topos, that means $i(f)$ is a local fibration in $\mathrm{Pre}(\mathcal{C})^{\Delta^{\mathrm{op}}}$. Since sheafification preserves local fibrations as well, $\ell(i(f)) \cong f$ must be a local fibration in $\mathcal{E}^{\Delta^{\mathrm{op}}}$.

**Corollary 1.5.** *$\mathcal{E}^{\Delta^{op}}$ possesses a weak factorization system $(\mathcal{A}, \mathcal{F})$ such that $\mathcal{A}$ is a subclass of $\mathsf{W}$, and $\mathcal{F}$ is a subclass of local fibrations.*

Indeed, for any locally presentable category $\mathcal{K}$ and set of morphisms $A$, the class of "$A$-cofibrations", that is, the saturation of $A$ under arbitrary pushouts, transfinite compositions and retracts, together with "$A$-fibrations", that is, the class of morphisms with the right lifting property against members of $A$, will form a weak factorization system. (This is the "small object argument"; weaker properties of $\mathcal{K}$ than local presentability suffice to establish it. See [2] for details.) Apply the small object argument to $\mathcal{K} = \mathcal{E}^{\Delta^{\mathrm{op}}}$ and the set $A$ found in 1.4. □

We are now in a position to define $\phi$. Fix any weak factorization system $(\mathcal{A}, \mathcal{F})$ as in 1.5. Given a connected component of $\mathsf{frac}_{\mathrm{LAFib}^{-1}\mathcal{E}_F^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$, pick an object $\mathbf{1} \xleftarrow{w} C \xrightarrow{f} X$ from it, and factor $f$ as $pa$ with $a \in \mathcal{A}$, $p \in \mathcal{F}$. From the diagram

$$\begin{tikzcd} \mathbf{1} & C \arrow[l, "w"'] \arrow[r, "f"] \arrow[dr, tail, "a"] & X \\ & & \widetilde{C} \arrow[ull, "\sim"] \arrow[u, two heads, "p"'] \end{tikzcd}$$

one sees that the map marked $\sim$ is a weak equivalence, so $\mathbf{1} \xleftarrow{\sim} \widetilde{C} \xrightarrow{p} X$ is a simplicial torsor. The value of $\phi$ is the connected component of $\mathsf{ST}(X)$ containing $\mathbf{1} \xleftarrow{\sim} \widetilde{C} \xrightarrow{p} X$.

Two different factorizations $p_1 a_1$, $p_2 a_2$ of the same $f$ will lead to the same value:

$$\begin{tikzcd}
 & & \widetilde{C}_1 \arrow[lld, "\sim"'] \arrow[rd, "p_1"] & \\
\mathbf{1} & C \arrow[l, "\sim"] \arrow[ru, tail, "a_1"] \arrow[rd, tail, "a_2"'] \arrow[rr, "f"] & & X \\
 & & \widetilde{C}_2 \arrow[llu, "\sim"] \arrow[ru, "p_2"'] \arrow[uu, dotted] &
\end{tikzcd}$$

since the dotted filler exists by the lifting property.

The value of $\phi$ does not depend on the initial choice of representative from the given connected component of $\mathsf{frac}_{\mathrm{LAFib}^{-1}\mathcal{E}_F^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$ either: given $\mathbf{1} \xleftarrow{w_1} C_1 \xrightarrow{f_1} X$ and $\mathbf{1} \xleftarrow{w_2} C_2 \xrightarrow{f_2} X$ that are "adjacent" via $c$,

$$\begin{tikzcd}
 & C_2 \arrow[ld, "w_2"'] \arrow[r, tail] \arrow[rrd, "f_2"] & \widetilde{C}_2 \arrow[rd, "p_2"] & \\
\mathbf{1} & & & X \\
 & C_1 \arrow[lu, "w_1"] \arrow[uu, "c"] \arrow[r, tail, "a_1"'] \arrow[rru, "f_1"] & \widetilde{C}_1 \arrow[uu, dotted] \arrow[ru] &
\end{tikzcd}$$

the existence of the dotted lift shows that the respective factorizations will belong to the same connected component of $\mathsf{ST}(X)$. Finally, using a factorization of $f$ again one shows that every connected component of $\mathsf{ST}(X)$ contains a torsor $C \xrightarrow{p} X$ with $p \in \mathcal{F}$, and this implies that $\phi$ is onto. $\square$

*Remark* 1.6. The terminal object $\mathbf{1}$ may seem to play a tautologous role, but I prefer to keep it in the notation as a reminder that the argument in fact concerns the representation of global homotopy classes $\mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(U, X)$ by connected components of an indexing category. The source $U$ cannot be arbitrary, however. See section 3 for details.

## 2. Theme and variations

The reader has no doubt observed that in constructing the mapping $\phi$ of Prop. 3, one never used that the structure map $w$ of the initially given fraction $\mathbf{1} \xleftarrow{w} C \xrightarrow{f} X$ was a local fibration (in addition to being a weak equivalence). Indeed, there are various combinations of features that one can put on these structure maps.

**Notation 2.1.** In addition to $\mathsf{ST}(X)$, we will define four full subcategories of $\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$.

- Let $\mathsf{ST}^J(X)$ contain those fractions $\mathbf{1} \xleftarrow{w} C \xrightarrow{f} X$ where $f$ is a global fibration in the sense of Joyal, i.e. has the right lifting property with respect to all acyclic monomorphisms.

- Let $\mathsf{ST}^{\mathcal{F}}(X)$ contain the fractions $\mathbf{1} \xleftarrow{w} C \xrightarrow{f} X$ with $f \in \mathcal{F}$, where $\mathcal{F}$ is a subclass of local fibrations that arises via Cor. 1.5 to Lemma 1.4.
- Let $\mathsf{ST}^{V}(X)$ contain the fractions with $w$ an acyclic local fibration, $f$ arbitrary.
- Write $\mathsf{ST}^{W}(X)$ for $\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$ itself.

For a locally fibrant $X$, one has full inclusions

$$(*) \qquad \mathsf{ST}^{J}(X) \hookrightarrow \mathsf{ST}^{\mathcal{F}}(X) \hookrightarrow \mathsf{ST}(X) \hookrightarrow \mathsf{ST}^{V}(X) \hookrightarrow \mathsf{ST}^{W}(X)$$

**Proposition 2.2.** *Passing to connected components,* $(*)$ *becomes a string of bijections*

$$\pi_0\mathsf{ST}^{J}(X) = \pi_0\mathsf{ST}^{\mathcal{F}}(X) = \pi_0\mathsf{ST}(X) = \pi_0\mathsf{ST}^{V}(X) = \pi_0\mathsf{ST}^{W}(X) = \mathsf{ho}_{\mathcal{E}^{\Delta^{op}}}(\mathbf{1}, X)$$

Indeed, the map $\pi_0\mathsf{ST}^{J}(X) \xrightarrow{\gamma_J} \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$ that is the restriction of the $\gamma$ of Prop. 1 factors through all the inclusions. Use now Joyal's theorem that ⟨acyclic monomorphisms, global fibrations⟩ is a weak factorization system in $\mathcal{E}^{\Delta^{\mathrm{op}}}$ and the argument of Prop. 3 to show that $\gamma_J$ is a bijection. The existence of ⟨weak equivalence, global fibration⟩ resp. ⟨weak equivalence, $\mathcal{F}$⟩ resp. ⟨weak equivalence, local fibration⟩ factorizations — even in the absence of global lifting properties — show that the induced intermediate maps on connected components are surjective, establishing the proposition. □

*Remark* 2.3. The superscript $V$ stands for Verdier; see below. The letter $W$ is in deference to Wraith [16] who, in joint work with Joyal, introduced the category $\mathsf{ST}^{W}(X)$ (more precisely, its classifying topos) and observed that connected components of $\mathsf{ST}^{W}(K(A,n))$ biject with $H^n(\mathcal{E}, A)$ for a constant abelian group $A$. See also Joyal–Wraith [14]. In the non-abelian case, Prop. 2.2 does not seem to have been observed before.

The next page is a rapid survey of ideas that will be explored elsewhere.

**Hypercovers.** A *hypercover* of a topos $\mathcal{E}$ is usually understood to be a simplicial object $C$ in $\mathcal{E}$ that is locally fibrant and acyclic, i.e. such that the canonical $C \to \mathbf{1}$ is both a local fibration and a local weak equivalence (see Moerdijk [15]), or sometimes a simplicial object that is merely acyclic (see K. Brown [3]), or sometimes as a $C$ of the above types considered as an object in the category denoted $\pi$ below, which is the quotient of $\mathcal{E}^{\Delta^{\mathrm{op}}}$ by simplicial homotopy; see Jardine [9]. Verdier's original definition in the Appendix to SGA4 Exposé V (where he ascribes the idea of hypercovers to P. Cartier) is more restrictive, and makes use of a site of definition of $\mathcal{E}$. He also introduces truncated variants of hypercovers, however, that provide a beautiful interpolation between Čech cohomology and what is known as Čech-Verdier cohomology.

Write $\pi$ for the category obtained from $\mathcal{E}^{\Delta^{\mathrm{op}}}$ by quotienting the hom-sets by the equivalence relation generated by simplicial homotopy (the constant 1-simplex $\Delta[1]$ playing the role of the interval). Write $HC$ for the opposite of the diagram that is the image in $\pi$ of hypercovers of $\mathcal{E}$ (up to equivalence, it doesn't matter which definition one chooses). $HC$ is a large, filtered diagram that possesses (non-canonically in $\mathcal{E}$) small cofinal subdiagrams. The homotopical version of Verdier cohomology can then be phrased as the existence of canonical bijections between $\operatorname*{colim}_{C\in HC} \pi(C,X)$, $\mathsf{ST}^V(X)$ and $\mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1},X)$ for any locally fibrant $X$.

**Combinatorial torsors.** Under specializing $X$ to be the nerve of a group $G$, none of the torsors listed in Prop. 2.2 bears the vaguest resemblance to principal homogeneous $G$-objects. Indeed, any simplicial torsor over $X$ possesses countably many independent pieces of structural data (the objects of $n$-simplices and the face and degeneracy maps) subject to infinitely many constraints (local acyclicity and fibrancy for the local torsors, and lifting properties for the global ones). This is very different from the notion of "principal homogeneous $G$-space", which is just a $G$-object subject to two diagrammatic constraints. Nonetheless, these more rigid and constrained torsors form full subcategories of the simplicial ones. They can be introduced only for bases $X$ that enjoy distinguished simplicial geometries: for example, $X$ that is $n$-coskeletal, or an $n$-hypergroupoid in the sense of Duskin, or an Eilenberg–MacLane object. Over an $n$-coskeletal $X$, one can talk of simplicial torsors $C \xrightarrow{f} X$ such that the map $f$ is $n$-coskeletal, or over an $n$-hypergroupoid $X$ one can restrict to simplicial torsors $C \xrightarrow{f} X$ such that $f$ is an exact fibration above dimension $n$, and over Eilenberg–MacLane objects Duskin has defined $K(A,n)$-torsors. These varieties are determined by a finite number of simplicial levels and structure maps. Once one has carved out a subcategory of combinatorial torsors $\mathsf{ST}^C(X) \hookrightarrow \mathsf{ST}(X)$, the goal is to verify that the inclusion induces a bijection on $\pi_0$. This is easiest shown with the help of a "$\pi_0$-deformation retraction" of $\mathsf{ST}(X)$ to $\mathsf{ST}^C(X)$. That is, one finds a functor $\mathsf{ST}(X) \xrightarrow{R} \mathsf{ST}^C(X)$ and a natural transformation $\Phi$ of zig-zags of fixed shape (say, of length three, for illustration) $\Phi(T) = T \leftarrow \bullet \rightarrow \bullet \leftarrow R(T)$ linking the torsors $T$ and $R(T)$ in $\mathsf{ST}(X)$. The existence of such a zig-zag, starting with an arbitrary $T \in \mathsf{ST}(X)$, shows $\pi_0\mathsf{ST}^C(X) \to \pi_0\mathsf{ST}(X)$ onto. If $\Phi$ is functorial and has the property that $\Phi(T)$ is a zig-zag of combinatorial torsors for any $T \in \mathsf{ST}^C(X)$, then $\pi_0\mathsf{ST}^C(X) \to \pi_0\mathsf{ST}(X)$ is into. (Note that the simplest example of such $(R,\Phi)$ is a functorial factorization of simplicial torsors by combinatorial ones.) If the species of torsors considered are definable in geometric logic, so they have classifying topoi, then the data $(R,\Phi)$ amount to precisely the same as a "natural homotopy inverse" in the sense of Joyal and Wraith [14] to the geometric morphism $\mathbb{B}[\mathsf{ST}^C] \hookrightarrow \mathbb{B}[\mathsf{ST}]$ induced by the fact that combinatorial torsors form a quotient theory of simplicial ones.

**Algebraic operations on torsors.** All five types of torsors considered in Prop. 2.2 are preserved by products, giving rise to functors $\mathsf{ST}(X) \times \mathsf{ST}(Y) \to \mathsf{ST}(X \times Y)$. These pass to maps $\pi_0\mathsf{ST}(X) \times \pi_0\mathsf{ST}(Y) \to \pi_0\mathsf{ST}(X \times Y)$. When $X$ is an (abelian) group or monoid in $\mathcal{E}^{\Delta^{\mathrm{op}}}$, they give the algebraic structure on homotopy classes. It is also possible to describe the effect of changing the base topos. Verdier, Wraith and simplicial torsors are of a local nature, thus preserved by inverse image parts of geometric morphisms. For clarity, write $\mathsf{ST}_{\mathcal{E}}(X)$ for the category of simplicial torsors in $\mathcal{E}^{\Delta^{\mathrm{op}}}$ over $X$. If $\mathcal{E}$, $\mathcal{F}$ are topoi and $\mathcal{E} \times \mathcal{F}$ stands for their product in the category of Grothendieck topoi and geometric morphisms, with $\mathrm{pr}_i$ $(i = 1,2)$ being the projections to the factors, then there is a functor $\mathsf{ST}_{\mathcal{E}}(X) \times \mathsf{ST}_{\mathcal{F}}(Y) \to \mathsf{ST}_{\mathcal{E}\times\mathcal{F}}(\mathrm{pr}_1^*(X) \times \mathrm{pr}_2^*(Y))$ that can be used to construct the external product in cohomology.

## 3. Calculus of zig-zags and fractions

Let now $\mathcal{M}$ be any category with a distinguished subcategory $\mathsf{W}$, which we will assume to contain all identity morphisms of $\mathcal{M}$. Maps in $\mathsf{W}$ will be referred to as *weak equivalences*, and we'll still write $\mathsf{ho}_{\mathcal{M}}$ for $\mathcal{M}[\mathsf{W}^{-1}]$. Dwyer–Kan [5], [6] provide a substantial generalization of the Gabriel–Zisman [7] theory of localizations of categories, a key aspect being that $\mathsf{ho}_{\mathcal{M}}$ is now viewed as a category enriched over *SSet* via "homotopical function complexes". We will ignore here (for the time being) the higher-dimensional simplices, concentrating on the dimension 0 part, i.e. on the morphisms of $\mathsf{ho}_{\mathcal{M}}$. We begin with a symmetrized version of fractions due to Dwyer and Kan.

**Definition 3.1.** For objects $U, X$ of $\mathcal{M}$, let $\mathsf{zig}(U,X)$ be the category whose objects are zig-zags of length 3, $U \xleftarrow{w_1} C \xrightarrow{f} F \xleftarrow{w_2} X$ where $w_1, w_2$ are weak equivalences; $C, F, f$ are arbitrary. Let a morphism from $U \leftarrow C_1 \to F_1 \leftarrow X$ to $U \leftarrow C_2 \to F_2 \leftarrow X$ be maps $C_1 \to C_2$, $F_1 \to F_2$ that make

$$\begin{tikzcd}
& C_1 \arrow[r] \arrow[dl] \arrow[dd] & F_1 \arrow[dd] & \\
U & & & X \arrow[ul] \arrow[dl] \\
& C_2 \arrow[r] \arrow[ul] & F_2 &
\end{tikzcd}$$

commute.

The localization functor $\mathcal{M} \xrightarrow{\mathsf{ho}} \mathsf{ho}_{\mathcal{M}}$ induces a map (of classes, in general) $\pi_0\mathsf{zig}(U,X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U,X)$ by sending a representative $U \xleftarrow{w_1} C \xrightarrow{f} F \xleftarrow{w_2} X$ of a connected component to $U \xrightarrow{(\mathsf{ho}w_1)^{-1}} C \xrightarrow{\mathsf{ho}f} F \xrightarrow{(\mathsf{ho}w_2)^{-1}} X$; this is well-defined, since if the left-hand diagram commutes, so does the one on the right:

$$\begin{tikzcd}
& C_1 \arrow[r, "f_1"] \arrow[dl, "w_1"'] \arrow[dd, "c"] & F_1 \arrow[dd, "f"] & \\
U & & & X \arrow[ul, "w_2"'] \arrow[dl, "u_2"] \\
& C_2 \arrow[r, "f_2"'] \arrow[ul, "u_1"] & F_2 &
\end{tikzcd}
\quad \overset{\gamma}{\Longmapsto} \quad
\begin{tikzcd}
& C_1 \arrow[r, "\mathsf{ho}f_1"] \arrow[dd, "\mathsf{ho}c"'] & F_1 \arrow[dd, "\mathsf{ho}f"] \arrow[dr, "(\mathsf{ho}w_2)^{-1}"] & \\
U \arrow[ur, "(\mathsf{ho}w_1)^{-1}"] \arrow[dr, "(\mathsf{ho}u_1)^{-1}"'] & & & X \\
& C_2 \arrow[r, "\mathsf{ho}f_2"'] & F_2 \arrow[ur, "(\mathsf{ho}u_2)^{-1}"'] &
\end{tikzcd}$$

**Theorem 3.2.** (Dwyer–Kan) *Let $\mathcal{M}, \mathsf{W}$ form part of a Quillen model category. The mapping $\pi_0\mathsf{zig}(U,X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U,X)$ is a bijection (of sets) for any $U, X \in \mathcal{M}$.*

For a Quillen model category with functorial factorizations of maps into an (acyclic) cofibration followed by an acyclic fibration (resp. fibration), the proof is sketched in Dwyer–Kan [5]. For the general case, see Dwyer–Kan [6].

Rather than some kind of generalized torsor, it is useful to think of a short zig-zag *as if* it were a map from a cofibrant approximation to $U$ to a fibrant approximation to $X$, thus an approximation to a class in $\mathsf{ho}(U,X)$ in Quillen's sense. The remarkable feature of this case of the "hammock localization" is that no explicit preferred class of (co)fibrations enters, though the existence of *some* such is needed in the proof — to define composition of zig-zags, in fact, whence it is easy to prove that $\mathsf{zig}$ has the requisite universal property.

The category of right fractions $\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{M}}(U,X)$, as well as $\mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U,X)$, the category of left fractions, are full subcategories of $\mathsf{zig}(U,X)$. They consist of the zig-zags for which $w_2$ resp. $w_1$ is the identity. What is still open, I believe, is the deceptively simple-looking

**Problem 3.3.** Find sufficient and necessary conditions on the pair $(\mathcal{M},\mathsf{W})$ for

$$\pi_0\mathsf{zig}(U,X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U,X)$$

resp.

$$\pi_0\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{M}}(U,X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U,X)$$

resp.

$$\pi_0\mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U,X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U,X)$$

to be bijections for arbitrary $U$, $X$.

Let us now give a name to the condition introduced in Prop. 2. Both for variety and for the sake of applications, we treat its category-theoretic dual here.

**Definition 3.4.** Given a diagram $\mathcal{F} := U \xleftarrow{w} C \xrightarrow{f} X$, $w \in \mathsf{W}$, let the corresponding *category of amalgamations*, $\mathsf{amal}_{\mathcal{F}}$, have as objects all commutative diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "w"'] & X \arrow[d, "u"] \\ U \arrow[r, "g"'] & F \end{tikzcd}$$

with $u \in \mathsf{W}$, a morphism between two such being an arrow $F_1 \xrightarrow{h} F_2$ that makes

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "w"'] & X \arrow[d, "u_1"'] \arrow[ddr, "u_2"] & \\ U \arrow[r, "g_1"] \arrow[drr, "g_2"'] & F_1 \arrow[dr, "h"] & \\ & & F_2 \end{tikzcd}$$

commute (note $u_1, u_2, w \in \mathsf{W}$). Say that $(\mathcal{M},\mathsf{W})$ possess a *connected calculus of left fractions* if for any fraction $\mathcal{F}$, $\mathsf{amal}_{\mathcal{F}}$ is non-empty and connected.

An argument mirroring Prop. 1 shows

**Proposition 3.5.** *If* $(\mathcal{M},\mathsf{W})$ *possess a connected calculus of left fractions, then the localization map* $\pi_0\mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U,X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U,X)$ *is a bijection for arbitrary* $U$, $X$.

The Gabriel–Zisman axioms for a left calculus of fractions state that $\mathsf{amal}_{\mathcal{F}}$ is non-empty, and for every commutative diagram in $\mathcal{M}$ of the form $\bullet \xrightarrow{u} \bullet \underset{g}{\overset{f}{\rightrightarrows}} \bullet$ with $u \in \mathsf{W}$, there exists $v \in \mathsf{W}$ making $\bullet \underset{g}{\overset{f}{\rightrightarrows}} \bullet \xrightarrow{v} \bullet$ commute. Manipulations with fractions that are tedious to typeset but easy to perform establish

**Proposition 3.6.** *If $(\mathcal{M}, \mathsf{W})$ satisfy the Gabriel–Zisman axioms for a left calculus of fractions, then each $\mathsf{amal}_{\mathcal{F}}$ is a filtered category. A fortiori it is non-empty and connected, i.e. $(\mathcal{M}, \mathsf{W})$ possess a connected calculus of left fractions.*

*Remark* 3.7. Dually to the last observation in Prop. 1, if the class $\mathsf{W}$ is closed under pushout in $\mathcal{M}$, then each $\mathsf{amal}_{\mathcal{F}}$ has an initial object and is therefore non-empty and connected. Such $(\mathcal{M}, \mathsf{W})$ are typical examples of connected but not Gabriel–Zisman calculi.

**Applications of the connected calculus.** The goal is to show that the connected calculus provides a sufficient but not necessary condition for the localization map from fractions to homotopy classes to be bijective. En route, however, we find a "fractional" expression for global homotopy classes that is valid between arbitrary simplicial objects in a topos. (The categories considered from here on will all be locally small.)

An argument dual to Prop. 2 in the Quillen-axiomatic setting establishes

**Proposition 3.8.** *Let $\mathcal{M}$ be a Quillen model category, and* AcCof *the subcategory of acyclic cofibrations. Then $\pi_0 \mathsf{frac}_{\mathcal{M}\,\mathrm{AcCof}^{-1}}(U, X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{M}}(U, X)$ is a bijection if both $U$ and $X$ are cofibrant.*

Analogously to the case of Wraith's torsors $\mathsf{ST}^W$, it is possible to "clear the cofibrations from the denominators".

**Proposition 3.9.** *In a Quillen model category $\mathcal{M}$, the localization map $\pi_0 \mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U, X) \xrightarrow{\iota} \mathsf{ho}_{\mathcal{E}^{\Delta^{op}}}(U, X)$ is a bijection provided $U$, $X$ are cofibrant.*

Consider the commutative diagram of sets and mappings

$$\begin{tikzcd}
\pi_0 \mathsf{frac}_{\mathcal{M}\,\mathrm{AcCof}^{-1}}(U, X) \arrow[rr, two heads, "\phi"] \arrow[dr, "\iota"'] & & \pi_0 \mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U, X) \arrow[dl] \\
& \mathsf{ho}_{\mathcal{M}}(U, X) &
\end{tikzcd}$$

where $\phi$ is now induced by the inclusion $\mathsf{frac}_{\mathcal{M}\,\mathrm{AcCof}^{-1}}(U, X) \hookrightarrow \mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U, X)$. It is enough to show $\phi$ surjective. Take any $U \to C \xrightarrow{w} X$ with $w \in \mathsf{W}$, and factor it as a cofibration $c$

followed by an acyclic fibration $f$:

$$\begin{tikzcd} & \widetilde{C} \arrow[dd, two heads, "f"] & \\ U \arrow[ur, dotted] \arrow[dr] & & X \arrow[ul, "c"'] \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

The 2-of-3 property of weak equivalences forces $c$ to be an acyclic cofibration, and the dotted lift exists since $U$ was assumed cofibrant. This shows that all connected components of $\mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U, X)$ are hit by $\phi$. $\square$

Since the default Quillen model structure on simplicial objects in a topos has all objects cofibrant, one obtains

**Corollary 3.10.** *The localization map* $\pi_0 \mathsf{frac}_{\mathcal{E}^{\Delta^{op}}\mathsf{W}^{-1}}(U, X) \to \mathsf{ho}_{\mathcal{E}^{\Delta^{op}}}(U, X)$ *is a bijection for any* $U, X$.

It is not the case, however, that $(\mathcal{E}^{\Delta^{\rm op}}, \mathsf{W})$ satisfies even a connected calculus of left fractions. I am indebted to Phil Hirschhorn for an explicit example of a fraction $\mathcal{F} = C \xleftarrow{w} A \xrightarrow{f} B$ in $(SSet, \mathsf{W})$ such that $\mathsf{amal}_{\mathcal{F}}$ is empty.

**Example 3.11.** (Phil Hirschhorn) Let $A$ be the union of two 1-simplices, named $a_1$ and $a_2$, with their endpoints identified so that the realization is homeomorphic to a circle. Let $B$ be a 1-simplex named $b$ with its endpoints identified so that its realization is homeomorphic to a circle; and let $C$ be a 1-simplex $c$ with its endpoints identified so that its realization is homeomorphic to a circle.

Let the map $f$ collapse $a_2$ to the vertex of $B$, with $a_1$ going to $b$. This is a weak equivalence (although it doesn't have to be to make $\mathcal{F}$ a valid fraction). Let the map $w$ collapse $a_1$ to the vertex of $C$, with $a_2$ going to $c$. This map too is a weak equivalence.

Suppose there was a way to amalgamate the diagram to a square using some simplicial set $D$. Since $A \to B$ and $A \to C$ are weak equivalences, and $B \to D$ is to be a weak equivalence, therefore so would $C \to D$ have to be.

However, if the diagram is to commute on the nose (i.e. not just up to homotopy), then the 1-simplex $c$ of $C$ must go to a single vertex of $D$, which means that the induced homomorphism on the fundamental group is trivial.

*Remark* 3.12. Let $\mathcal{M}$ be a Quillen model category with all objects cofibrant. It is easy to show that for any fraction $\mathcal{F} = C \xleftarrow{w} A \xrightarrow{f} B$, any two objects of $\mathsf{amal}_{\mathcal{F}}$ will be connected by a zig-zag of length three. (The argument doesn't even need $w \in \mathsf{W}$.) So the real obstruction is $\mathsf{amal}_{\mathcal{F}}$ being empty.

*Remark* 3.13. Dwyer–Kan [5] introduce conditions under the names of *left* resp. *right* resp. *two-sided homotopy calculus of fractions* that imply that the localization maps from the corresponding categories of fractions are bijections. (They obtain conclusions actually stronger than that sought by Problem 3.3, namely weak equivalences between homotopy function complexes in their sense.) These calculi are based on $(\mathcal{M}, \mathsf{W})$ alone (without reference to Quillen model categories), but their definitions seem hard to check in practice. Dwyer and Kan state, however, that (for a homotopy calculus of left fractions, to take an example) it suffices that $\mathsf{W}$ satisfies the 2-of-3 property, and *functorial* completions of fractions to commutative squares exist; or that $\mathsf{W}$ is closed under pushout. The notion of connected calculus of fractions (which probably exists in the literature already, though it was suggested to me only implicitly by Wraith's papers and the hypercovers of Verdier and K. Brown) must be closely connected. It is probably just the $\pi_0$ level of a more powerful notion. One could demand, for example, that the nerve of $\mathsf{amal}_{\mathcal{F}}$ have the weak homotopy type of a point for any fraction $\mathcal{F}$, and see what this implies about the localization functors enriched simplicially.

In the direction of *weakening* the connected calculus of fractions, one could also consider an up-to-homotopy version, where the squares (amalgams) are only required to commute in a category gotten from $\mathcal{M}$ by quotienting out an equivalence relation ("elementary homotopies") on the hom-sets. This presumes that cylinder or path objects exist, so there's more of a homotopy model structure on $\mathcal{M}$ than just the bare subcategory of weak equivalences.

*Remark* 3.14. Quillen's axioms are self-dual, but it is far from the case that the known homotopy model structures on $\mathcal{E}^{\Delta^{\mathrm{op}}}$ are symmetric. Under Joyal's definitions, every object is cofibrant, and the class of acyclic cofibrations is definable in geometric logic (a fortiori, is of local nature) while neither of these holds for global fibrations. So it is natural to ask, dually to Cor. 3.10,

**Question 3.15.** What is the largest subcategory of $\mathcal{E}^{\Delta^{\mathrm{op}}}$ restricted to which the localization map $\pi_0\mathsf{frac}_{\mathsf{W}^{-1}\mathcal{E}^{\Delta^{\mathrm{op}}}}(U, X) \xrightarrow{\gamma} \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(U, X)$ is a bijection?

By the usual factorization arguments, $\gamma$ is bijective when (i) $U$ is locally and $X$ globally fibrant, and also when (ii) $U$ is the terminal object and $X$ is locally fibrant. It would be very pleasing if the answer turned out to be the class of locally fibrant objects, which are "malleable" and can be constructed by explicit finitary means — rather unlike globally fibrant ones. Note that a Quillen model category, by default, only comes equipped with one notion of fibrancy, and the fact that the axioms are typically proved with the help of a functorial but transfinite factorization construction (the small object argument) means that fibrant replacement is uniform, but uniformly inefficient; it is as if in the toolbox of group cohomology, only the bar resolution existed. Abelian sheaf cohomology is largely made computable by flabby and soft resolutions, and more generally by acyclic models, for which no clear analogue is known in non-abelian Quillen model categories over sheaves. See Beke [1] for further results on the curious indeterminacy in the choice of fibrations for a fixed category of models and subcategory of weak equivalences.

**How long a zig-zag?** $\pi_0\mathcal{C}$ is the collapse of the class of objects of $\mathcal{C}$ under an equivalence relation that is describable (a priori) only as the saturation of the relation "there is a morphism between". Effective use of this notion would be quite hopeless if there was no upper bound on the length of iterations, or length of zig-zags, needed to connect two objects that are in the same component of $\mathcal{C}$. It is a well-known feature of Yoneda's Ext theory that if two $n$-fold extensions are connected by a zig-zag "ladder" of extensions, then they are connected by a two-tier one. The thesis of Glenn [8] contains a purely simplicial proof of this — in fact, an extension to simplicial $n$-hypergroupoids in an arbitrary exact category. As a coda, let us point out that the existence of short bridges is a purely formal consequence of the axioms of the connected calculus, and "one third" of Quillen's 2-of-3 axiom for weak equivalences.

**Proposition 3.16.** *Let $\mathcal{M}$ be a category with subcategory $\mathsf{W}$. Suppose that*

(i) *for any right fraction $\mathcal{F}$, $\mathsf{amal}_{\mathcal{F}}$ is non-empty*
(ii) *if the composite $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \in \mathsf{W}$ and $f \in \mathsf{W}$, then $g \in \mathsf{W}$.*

*(a) Then if two fractions belong to the same connected component of $\mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U, X)$, then they are already connected by a zig-zag of shape $\bullet \to \bullet \leftarrow \bullet$.*

*(b) Suppose, in addition, that all $\mathsf{amal}_{\mathcal{F}}$ are connected, i.e. that $(\mathcal{M}, \mathsf{W})$ possess a connected calculus of left fractions. Then any two objects of any $\mathsf{amal}_{\mathcal{F}}$ are already connected by a zig-zag of shape $\bullet \to \bullet \leftarrow \bullet$.*

*Proof.* $(a)$ Let the original fractions be $U \to C_1 \xleftarrow{w_1} X$ and $U \to C_2 \xleftarrow{w_2} X$, and suppose they are connected by a zig-zag of shape $\bullet \leftarrow \bullet \to \bullet$ in $\mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U, X)$, the fraction mapping to both being $U \to C \leftarrow X$:

$$
\begin{tikzcd}
 & A & C_1 \arrow[l] & & \\
U \arrow[ur] \arrow[urr] \arrow[rrr] \arrow[drr] & & & C \arrow[ul] \arrow[dl] & X \arrow[l] \arrow[ull, "w_1"'] \arrow[dll, "w_2"] \\
 & & C_2 \arrow[uul] & &
\end{tikzcd}
$$

By condition $(ii)$, $C \to C_1 \in \mathsf{W}$. By condition $(i)$, there is an amalgamation of $C_2 \leftarrow C \to C_1$ to an $A$ such that $C_2 \to A \in \mathsf{W}$. So the composite $A \leftarrow C_2 \leftarrow X \in \mathsf{W}$, and the fraction $U \to A \leftarrow X$ receives maps from both of the original ones. The claim now follows by iteratively reducing the length of the zig-zag that is assumed to connect the two objects of $\mathsf{frac}_{\mathcal{M}\mathsf{W}^{-1}}(U, X)$.

The proof of $(b)$ is similar. $\square$

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