Accessible functors, λ-equivalent objects, and the Lefschetz Principle

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Lefschetz Principle à la Lefschetz

" In the present appendix we propose to show that, in a certain sense, algebraic geometry over a groundfield of characteristic zero may be reduced to complex algebraic geometry. This is without question the deep reason why characteristic zero algebraic geometry presents no new results over and above complex algebraic geometry. "

Lefschetz, *Algebraic Geometry*, 1953

Lefschetz states that this follows from the fact that any variety is determined by a finite tuple of coefficients, that finitely generated fields of characteristic zero can be embedded into $\mathbb{C}$, and field embeddings preserve algebraic relations.

Lefschetz Principle à la Weil

" For a given value of the characteristic p, every result, involving only a finite number of points and of varieties, which has been proved for some choice of the universal domain remains valid without restriction; there is but one algebraic geometry of characteristic p for each value of p, not one algebraic geometry for each choice of the universal domain. "

A formal proof of this principle would require "a formal metamathematical characterization of the type of proposition to which it applies; this would have to depend on the metamathematical, i.e. logical analysis of all our definitions."

Weil, *Foundations of Algebraic Geometry*, 1962

A "universal domain" is an algebraically closed field of infinite transcendence degree over its prime field.
(This definition is due to Weil.)

similar instances

Note that

- representation theory of finite groups over *any* algebraically closed field of characteristic zero is "the same"
- representation theory of finite groups over any algebraically closed field of a fixed positive characteristic is the same
- classification of finite-dimensional Lie algebras over *any* algebraically closed field of characteristic zero is the "same"

These algebraically closed fields need not be universal domains.

Since the time of Lefschetz and Weil, considerable success with "the metamathematics of definitions" in algebraic geometry.

- Meta-theorems asserting that suitable statements are either true for all algebraically closed fields of a given characteristic, or none. (Tarski, Robinson)

  The class of these "suitable statements" is rather limited.

- Meta-theorems asserting that suitable statements are either true over all universal domains of a given characteristic, or none. (Barwise–Eklof–Feferman)

  The class of these statements is much bigger, but the meta-theorem is hard to apply in practice.

Our goal: meta-theorem asserting that all universal domains of a given characteristic are interchangeable, that is easy to apply in practice.

some classical notions

Let $\mathcal{L}$ be a first order language and let $\mathcal{S}_1$, $\mathcal{S}_2$ be $\mathcal{L}$-structures. $\mathcal{S}_1$ and $\mathcal{S}_2$ are called *elementarily equivalent* if for every sentence ϕ in the language $\mathcal{L}$, $\mathcal{S}_1 \models \phi$ if and only if $\mathcal{S}_2 \models \phi$.

Let $h : \mathcal{S}_1 \to \mathcal{S}_2$ be a homomorphism of structures. h is an *elementary embedding* if for every formula $\phi(x_1, x_2, \ldots, x_n)$ (with the x_i as free variables) and every n-tuple $a_i \in \mathcal{S}_1$,

$$\mathcal{S}_1 \models \phi(a_1, a_2, \ldots, a_n) \text{ if } \mathcal{S}_2 \models \phi(h(a_1), h(a_2), \ldots, h(a_n))$$

This implies that h is an embedding of structures; that the 'if' above is an 'if and only if'; and that $\mathcal{S}_1$ and $\mathcal{S}_2$ are elementarily equivalent.

two classical examples

(Tarski) Let K_1 and K_2 be algebraically closed fields of the same characteristic; then they are elementarily equivalent in the language of $+, -, \times, 0, 1$.

(Robinson) Let K/k be a field extension between algebraically closed fields; then this is an elementary embedding.

Note that the Lefschetz Principle (as formulated by Weil) does not stipulate that an extension K/k of universal domains *induces an isomorphism* between "algebraic geometry over k" and "algebraic geometry over K". (Weil was not thinking functorially.)

the first-order Lefschetz Principle

Let Θ be a first order sentence in the signature $+, -, \times, 0, 1$ of rings. The following are equivalent:

- $\mathbb{C} \models \Theta$
- $K \models \Theta$ for every algebraically closed field K of characteristic zero
- there is an infinite set of primes P such that $\overline{\mathbb{F}}_p \models \Theta$ for all $p \in P$
- there is an integer p_0 such that $\overline{\mathbb{F}}_p \models \Theta$ for all $p \geqslant p_0$.

Not so many properties of varieties (let alone schemes) permit a formulation as first order(!) sentences in the language of rings.

Nonetheless, there are applications of the first order Lefschetz Principle, such as the Ax–Grothendieck theorem.

relating elementary equivalences and embeddings

Note that if the algebraically closed fields K_1 and K_2 are elementarily equivalent, they are connected from a common source by elementary embeddings: $K_1 \leftarrow k \rightarrow K_2$ where k is the algebraic closure of their (common) prime field. They are also connected to a common target by elementary embeddings: $K_1 \rightarrow K \leftarrow K_2$ where K is the algebraic closure of a compositum of K_1 and K_2.

A famous theorem of Keisler (whose proof used the GCH) and Shelah (without the GCH) asserts that two structures $\mathcal{S}_1$ and $\mathcal{S}_2$ are elementarily equivalent if and only if they have isomorphic ultrapowers:

$$\prod_{I/\mathcal{U}} \mathcal{S}_1 \approx \prod_{I/\mathcal{U}} \mathcal{S}_2$$

Since each diagonal $\mathcal{S}_i \to \prod_{I/\mathcal{U}} \mathcal{S}_i$ is an elementary embedding, two elementarily equivalent structures can always be linked by a zig-zag $\mathcal{S}_1 \to \bullet \leftarrow \mathcal{S}_2$ of elementary embeddings.

elementary chains

The Tarski–Vaught theorem states that if $\mathcal{S}$ is the union of a well-ordered, continuous chain of structures $\mathcal{S}_\alpha$, where each $\mathcal{S}_\alpha \subseteq \mathcal{S}_{\alpha+1}$ is an elementary embedding, then $\mathcal{S}_\alpha \subseteq \mathcal{S}$ is an elementary embedding for each α.

A much better way of saying this is the following: let str_S be the category of S-structures and homomorphisms (in some fixed first-order signature S), and let elem_S be the category of S-structures and elementary embeddings. Then the canonical inclusion

$$\mathit{elem}_S \rightarrowtail \mathit{str}_S$$

creates filtered colimits.

infinitary logic

Let κ and λ be regular cardinals, $\kappa \geqslant \lambda$. Formulas of the logic $\mathcal{L}_{\kappa,\lambda}$ are built from atomic formulas using the operations of

- negation
- conjunctions or disjunctions of less than κ formulas
- existential or universal quantification over a tuple of less than λ variables.

$\mathcal{L}_{\infty,\lambda}$ is the union of all the logics $\mathcal{L}_{\kappa,\lambda}$.

Elementary equivalence and elementary embedding are defined for $\mathcal{L}_{\infty,\lambda}$ in the expected way.

examples

- Finitely generated groups can be defined in $\mathcal{L}_{\omega_1\omega}$ in the language of groups
- The property of being path connected can be defined for real semi-algebraic sets in $\mathcal{L}_{\omega_1\omega}$, in the signature $+, -, \times, <, 0, 1$
- well-orders can be defined in $\mathcal{L}_{\omega_1\omega_1}$ (but not in $\mathcal{L}_{\infty,\omega}$) in the signature $<$.

Karp's theorem

Let $\mathcal{S}_1$, $\mathcal{S}_2$ be structures for a language in $\mathcal{L}_{\infty,\lambda}$. A partial isomorphism $\mathcal{S}_1 \leftarrowtail D \rightarrowtail \mathcal{S}_2$ is a structure D equipped with embeddings into $\mathcal{S}_1$, $\mathcal{S}_2$. A set I of partial isomorphisms $\mathcal{S}_1 \leftarrowtail D_i \rightarrowtail \mathcal{S}_2$ satisfies the "$< \lambda$ back and forth property" if

- for every $i \in I$ and set $X \subseteq \mathcal{S}_1$ with $|X| < \lambda$, there exists $j \in I$ such that $\mathcal{S}_1 \leftarrowtail D_j \rightarrowtail \mathcal{S}_2$ extends $\mathcal{S}_1 \leftarrowtail D_i \rightarrowtail \mathcal{S}_2$ and the image of D_j in $\mathcal{S}_1$ contains X, and
- for every $i \in I$ and set $Y \subseteq \mathcal{S}_2$ with $|Y| < \lambda$, there exists $j \in I$ such that $\mathcal{S}_1 \leftarrowtail D_j \rightarrowtail \mathcal{S}_2$ extends $\mathcal{S}_1 \leftarrowtail D_i \rightarrowtail \mathcal{S}_2$ and the image of D_j in $\mathcal{S}_2$ contains Y.

Karp's theorem

$\mathcal{S}_1$ and $\mathcal{S}_2$ are "$< \lambda$ back and forth equivalent" if there is a set of partial isomorphisms between them satisfying this property.

Theorem (Karp) $\mathcal{S}_1$ and $\mathcal{S}_2$ are $< \lambda$ back and forth equivalent if and only if they are $\mathcal{L}_{\infty,\lambda}$-equivalent.

There are variants for $\mathcal{L}_{\infty,\lambda}$-embeddings and for fragments of $\mathcal{L}_{\infty,\lambda}$ with prescribed quantifier prefixes.

No (easy) analogues for $\mathcal{L}_{\omega,\omega}$!

Let K and k be algebraically closed, of the same characteristic, both of cardinality $> \lambda$. Then

- K and k are $\mathcal{L}_{\infty,\lambda}$-equivalent
- any field extension K/k between them is a $\mathcal{L}_{\infty,\lambda}$-elementary embedding.

In particular, any two of Weil's "universal domains" of the same characteristic are $\mathcal{L}_{\infty,\omega}$-equivalent.

This suggests that Weil's version of the Lefschetz principle should read: algebraic geometric properties that can be formulated as sentences of $\mathcal{L}_{\infty,\omega}$ are either true over all universal domains of a given characteristic, or none.

But *what* are these properties? How can one recognize them in practice?

Hodges (several articles, 1973 – ca. 1990):

"word constructions" (transfinite compositions of extensions by definition, and definable quotients)

Main result: if $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\mathcal{L}_{\infty,\lambda}$-equivalent, and W a suitable word construction, then $W(\mathcal{S}_1)$ and $W(\mathcal{S}_2)$ are $\mathcal{L}_{\infty,\lambda}$-equivalent.

Feferman (1970), Eklof (1973):

If F is a "λ-local functor" from str_{S_1} to str_{S_2}, then F preserves $\mathcal{L}_{\infty,\lambda}$-equivalence.

ular theorem of algebraic geometry we are interested in. For example, let us look at Weil's example ([**6**, pp. 307ff.]) which we rewrite in a form which makes clearer its logical structure. The superscripts on variables name the sort of variable, i.e. $v^{(n)} \in A_n$ as defined above, e.g. $v^{(5)}$ is a variable standing for an abstract variety. For each $0 \leqq r \leqq n$,

$$
\begin{aligned}
&\forall v^{(5)}\Big[(\{v^{(5)} \text{ is complete and has no multiple points}\} \wedge \{\dim v^{(5)} = n\}) \\
&\quad \to \Big(\bigvee_m \exists v_1^{(6)} \cdots \exists v_m^{(6)}\Big[\bigwedge_{i=1}^{m} (\{v_i^{(6)} \text{ is a cycle on } v^{(5)}\} \wedge \{\dim v_i^{(6)} = r\}) \\
&\qquad \wedge \forall w_1^{(6)} \forall w_2^{(6)}\Big[\Big(\bigwedge_{i=1}^{2} \{w_i^{(6)} \text{ is a cycle on } v^{(5)}\} \\
&\qquad\qquad \wedge \{\dim w_1^{(6)} = r\} \wedge \{\dim w_2^{(6)} = n - r\} \\
&\qquad\qquad \wedge \{w_2^{(6)} \text{ intersects } w_1^{(6)} \text{ properly on } v^{(5)}\} \\
&\qquad\qquad \wedge \bigwedge_{i=1}^{m} \{w_2^{(6)} \text{ intersects } v_i^{(6)} \text{ properly on } v^{(5)}\}\Big) \\
&\qquad\quad \to \bigvee_{(a_1 \cdots a_m) \in \omega^m} \Big\{\deg(w_1^{(6)} \cdot w_2^{(6)}) = \sum_i a_i \deg(v_i^{(6)} \cdot w_2^{(6)})\Big\}\Big]\Big]\Big)\Big].
\end{aligned}
$$

The expressions in braces are relations which we include in the structure $\mathfrak{A} = F(U)$ where $U \in \mathscr{U}_0$. Then the above sentence is a sentence of the language $L_{\infty\omega}$ corresponding to $\mathfrak{A}$, and to be able to apply the theorem

$$S_1\text{-structures} \xrightarrow{\text{nice functor}} S_2\text{-structures}$$

$$\cancel{S_1\text{-structures}} \xrightarrow{\text{nice functor}} \cancel{S_2\text{-structures}}$$

nice category   nice category

goals

Create a calculus of "λ–equivalence" for objects in (suitable) categories that

- is applicable to a wide class of categories
- specializes to $\mathcal{L}_{\infty,\lambda}$-equivalence for categories of structures
- possesses all formal properties of $\mathcal{L}_{\infty,\lambda}$-equivalence, but can be formulated without reference to signature or language (using *just* a category as background)
- comes with a matching notion of "λ-embedding" that specializes to $\mathcal{L}_{\infty,\lambda}$-embedding
- extends the Feferman–Eklof theorem to a situation $\mathcal{C} \xrightarrow{F} \mathcal{D}$ where the categories $\mathcal{C}$, $\mathcal{D}$ need not be assumed to be categories of structures.

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Good news: this is 90% done (TB–Rosicky 2015).

mono-generated categories

Let $\mathcal{C}$ be a category and λ a regular cardinal.

- object $X \in \mathcal{C}$ is λ-*generated* if $\hom(X, -)$ preserves those λ-directed diagrams of monos that exist in $\mathcal{C}$
- $\mathcal{C}$ is λ-*mono-generated* if every object can be written as the colimit of a λ-directed diagram of monos and λ-generated objects.

Remark: no assumption that *all* λ-directed diagrams of monos have a colimit, and no assumption that there is a *set* of λ-generated objects whose λ-directed colimits generate all objects.

Logical implications

$$\begin{array}{ccccc} \text{AEC} & \longrightarrow & \text{accessible} & \longrightarrow & \text{mono-accessible} \\ & & \downarrow & & \downarrow \\ & & \text{class-accessible} & \longrightarrow & \text{mono-generated} \end{array}$$

none of which is reversible.

example

Let FreeAbMono be the subcategory of abelian groups with objects the free abelian groups and morphisms the injective homomorphisms.

FreeAbMono is finitely mono-generated. Indeed, any directed colimit of monos in FreeAbMono — if it exists! — is "standard" (computed on underlying sets). Any finitely generated free abelian group is finitely generated as an object of FreeAbMono. Any free abelian group is directed colimit of its finitely generated (free) subgroups.

Whether FreeAbMono is accessible or not depends on set theory.

spans, dense sets of spans, λ-equivalence

A *span* between $X, Y \in \mathcal{C}$ is $X \leftarrowtail U \rightarrowtail Y$ where the arrows are mono. A set of spans $X \leftarrowtail U_i \rightarrowtail Y$ is λ-*dense* if (0) it is non-empty and (1) for all $i \in I$ and monomorphism $X \leftarrowtail G$ with λ-generated G, there exist $j \in I$ and morphisms $G \to U_j$ and $U_i \to U_j$ such that

$$\begin{tikzcd}
 & U_j \arrow[dl] \arrow[dr] & \\
X & G \arrow[l] \arrow[u] & Y \\
 & U_i \arrow[ul] \arrow[ur] \arrow[uu, bend right] &
\end{tikzcd}$$

commutes and (2) the symmetric condition, with respect to "test objects" $G \rightarrowtail Y$ with λ-generated G, holds too.

$X \sim_\lambda Y$, i.e. X and Y are λ-*equivalent*, if there is a λ-dense set of spans between them.

λ-embeddings

A morphism $f : X \to Y$ is called a λ-*embedding* if there is a λ-dense set $\mathcal{S}$ of spans between X and Y such that for any monomorphism $g : G \rightarrowtail X$ with λ-generated G there exist $X \xleftarrow{u} U \xrightarrow{v} Y \in \mathcal{S}$ and $t : G \to U$ such that in the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& U \arrow[ul, "u"'] \arrow[ur, "v"] & \\
& G \arrow[uul, "g"] \arrow[u, "t"'] &
\end{tikzcd}$$

$ut = g$ and $vt = fg$.

(A bit unlikely sounding, perhaps, but motivated by a variant of Karp's theorem for $\mathcal{L}_{\infty,\lambda}$-embeddings. It works.)

as expected

Work in a λ-mono-generated category.

- $\sim_\lambda$ is an equivalence relation
- if $X \sim_\lambda Y$ and X is λ-generated, then X and Y are isomorphic
- any λ-embedding is a monomorphism
- if $f : X \to Y$ is a λ-embedding with X, Y λ-generated, then f is an isomorphism
- λ-embeddings are closed under composition
- if g, gf are λ-embeddings, so is f.

The proofs play with spans.

Tarski–Vaught type of results

Work in a λ-mono-generated category $\mathcal{A}$ and assume that colimits referred to exist. Let λ-*emb*$(\mathcal{A})$ denote the subcategory of $\mathcal{A}$ whose morphisms are λ-embeddings.

- λ-directed colimit of natural transformations consisting of λ-embeddings between diagrams of λ-embeddings, is itself a λ-embedding
- the colimit cocone of a λ-directed diagram of λ-embeddings consists of λ-embeddings
- the inclusion $\lambda\text{-}emb(\mathcal{A}) \rightarrowtail \mathcal{A}$ creates λ-directed colimits
- in a finitely mono-generated category, a transfinite composite of finitary embeddings is a finitary embedding.

The proofs play with spans a little longer.

consistency with $\mathcal{L}_{\infty,\lambda}$-equivalence

Let S be a language for $\mathcal{L}_{\infty,\lambda}$ and let str_S denote the category of S-structures and homomorphisms, while emb_S is the category of S-structures and embeddings. For S-structures X and Y, the following are equivalent:

(1) X and Y are $\mathcal{L}_{\infty\lambda}$-elementary equivalent

(2) there is a set of partial isomorphisms between X and Y satisfying the $< \lambda$ back-and-forth property

(3) $X \sim_\lambda Y$ in $emb(\Sigma)$

(4) $X \sim_\lambda Y$ in $str(\Sigma)$.

The equivalence of (3) and (4) is not quite trivial. Need to play with factorizations of spans.

consistency with $\mathcal{L}_{\infty,\lambda}$-embedding

For an embedding $f : X \to Y$ of S-structures the following are equivalent:

(1) f is an $\mathcal{L}_{\infty\lambda}$-elementary embedding

(2) there is a set I of partial isomorphisms between X and Y satisfying the $< \lambda$ back-and-forth property, and such that for every subset Z of X of cardinality less than λ there is $h \in I$ such that $f(z) = h(z)$ for every $z \in Z$

(3) f is a λ-embedding in emb_S

(4) f is a λ-embedding in str_S.

Let $\mathcal{A}$ be a λ-mono-generated category and $F : \mathcal{A} \to \mathcal{B}$ a functor preserving monomorphisms and λ-directed colimits of monomorphisms.

- If $X \sim_\lambda Y$ then $F(X) \sim_\lambda F(Y)$.
- If $f \in \mathcal{A}$ is a λ-embedding, so is $F(f)$.

These are easy to prove, and specialize to the main result of Feferman–Eklof.

Not so useful in practice, since functors preserving monos are rare.

basic universal theories

Let $\lambda \geqslant \kappa$ be regular cardinals. Let $\forall_{\lambda\kappa}$ denote the class of theories in $\mathcal{L}_{\infty\kappa}$ that can be axiomatized by a set of sentences of the form

$$\forall \mathbf{x}(\phi \implies \psi)$$

where ϕ and ψ are built from atomic formulas using conjunction and disjunction only, where $\bigwedge_{i \in I}$ is permitted for $|I| < \lambda$ only, while $\bigvee_{j \in J}$ is permitted for J of any cardinality.

If T is a $\forall_{\lambda\kappa}$ theory, let $Mod(T)$ denote the category of models and homomorphisms of T.

basic universal theories

- more or less, models of Horn theories in $\mathcal{L}_{\lambda,\kappa}$
- contains all infinitary equational varieties, and quasi-varieties
- if T is $\forall_{\lambda\kappa}$ then $Mod(T)$ is accessible with well-behaved image factorizations of morphisms
- possible to characterize intrinsically categories equivalent to $Mod(T)$ for some basic universal T, but does not seem worth the trouble to spell it out.

theorem

Let $\mathcal{A}$ be a λ-mono-generated category, $T \in \forall_{\lambda\kappa}$, and $F : \mathcal{A} \to \mathit{Mod}(T)$ a functor that takes λ-directed colimits of monos to colimits.

- $X \sim_\lambda Y$ implies $F(X) \sim_\lambda F(Y)$
- If $f \in \mathcal{A}$ is a λ-embedding, so is $F(f)$.

– extends a beautiful result of Eklof

– useful since λ-directed colimit preserving functors abound

– proof uses accessibility mixed with image factorizations

– can one weaken the hypotheses on the target category? (missing 10%!)

sample application

Let S_1, S_2 be languages for $\mathcal{L}_{\infty,\lambda}$ and let $E : Mod(S_1) \to Mod(S_2)$ be an equivalence of categories.

- If X and Y are $\mathcal{L}_{\infty,\lambda}$-equivalent S_1-structures, then $E(X)$ and $E(Y)$ are $\mathcal{L}_{\infty,\lambda}$-equivalent S_2-structures.
- If f is an $\mathcal{L}_{\infty,\lambda}$-elementary embedding of S_1-structures, then $E(f)$ is an $\mathcal{L}_{\infty,\lambda}$-elementary embedding of S_2-structures.

Seems hard to prove syntactically or with Karp's theorem, since the S_1-structure of X (even the underlying set) cannot be related to the S_2-structure of $E(X)$.

sample application

If two groups are $\mathcal{L}_{\infty,\lambda}$-equivalent then so are their abelianizations.

Not easy to prove syntactically, since abelianization is a quotient (without definable representatives).

Similar observation holds for any morphism $T_1 \to T_2$ between (essentially) algebraic theories: it induces an adjunction between the locally presentable categories $Mod(T_1)$ and $Mod(T_2)$, where both left and right adjoints preserve (filtered enough) colimits.

Lefschetz–Weil in action

Let Set_{mono} denote the category of sets and monomorphisms. Then $I \sim_\lambda J$ in Set_{mono} if and only if either $\mathrm{card}(I) = \mathrm{card}(J) < \lambda$, or both $\mathrm{card}(I) \geqslant \lambda$ and $\mathrm{card}(J) \geqslant \lambda$.

Let k be a field. Use functor taking I to the algebraic closure of the field $k(x_i \,|\, i \in I)$ to deduce that if K_1, K_2 are algebraically closed, of infinite transcendence degree over k, then $K_1 \sim_\omega K_2$ in Field/k.

Lefschetz–Weil in action

For field k, field extension K/k and variety X_k over k, let X_K denote its base change (pullback along $spec(K) \to spec(k)$). Let $H^n(-, \mathbb{Z}/l\mathbb{Z})$ denote étale cohomology with constant coefficients $\mathbb{Z}/l\mathbb{Z}$. Fix k, X_k, $n \in \mathbb{N}$ and prime l.

The functor from Field$/k$ to vector spaces taking K to $H^n(X_K, \mathbb{Z}/l\mathbb{Z})$ preserves directed colimits (tough!).

Corollary For any two universal domains (over k), $H^n(X_{K_1}, \mathbb{Z}/l\mathbb{Z}) \sim_\omega H^n(X_{K_2}, \mathbb{Z}/l\mathbb{Z})$ in the category of $\mathbb{Z}/l\mathbb{Z}$ vector spaces.

Corollary If $H^n(X_K, \mathbb{Z}/l\mathbb{Z})$ is finite dimensional for *one* universal domain, then it is of the same finite dimension for *any* universal domain K.

By analogous reasoning ...

For any morphism $K_1 \to K_2$ of universal domains, $H^n(X_{K_1}, \mathbb{Z}/l\mathbb{Z}) \to H^n(X_{K_2}, \mathbb{Z}/l\mathbb{Z})$ is an isomorphism.

to do

- Can one characterize functors preserving λ-equivalence?
- Analogue of the calculus of λ-equivalence where spans $\bullet \leftarrowtail \bullet \rightarrowtail \bullet$ are not required to consist of monos?
- Analogue of the calculus of λ-equivalence for 2-categories, where pseudo-limits ("homotopy limits and colimits") are much better behaved than ordinary limits?
- Characterization of λ-equivalence in terms of zig-zag of λ-embeddings?
- There are categories that, for each regular cardinal λ, posses only a *set* of λ-equivalence classes of objects. How is this related to Shelah's classification theory (of models in terms of cardinal invariants)?

and even more to do

- Understand "back-and-forth equivalence" as "local isomorphism" in the sense of a suitable Grothendieck topology.
- Are categories of structured objects in algebraic geometry (e.g. varieties, schemes, ringed spaces, algebraic spaces, possibly over a base) λ-mono-generated?
- What accounts for the (observed) invariance of algebraic geometry with respect to base extensions between algebraically closed fields (not just between universal domains)?