THE GROTHENDIECK RING OF VARIETIES AND OF THE THEORY OF ALGEBRAICALLY CLOSED FIELDS

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ABSTRACT. In each characteristic, there is a canonical homomorphism from the Grothendieck ring of varieties to the Grothendieck ring of sets definable in the theory of algebraically closed fields. We prove that this homomorphism is an isomorphism in characteristic zero. In positive characteristics, we exhibit specific elements in the kernel of the corresponding homomorphism of Grothendieck *semi*rings. The comparison of these two Grothendieck rings in positive characteristics seems to be an open question, related to the difficult problem of cancellativity of the Grothendieck semigroup of varieties.

1. INTRODUCTION

Of the many occurrences of Grothendieck rings in algebraic geometry, there are two closely related ones that are the subjects of this note. One is Grothendieck's original definition: the generators are isomorphism classes of varieties, and the relations stem from open-closed decompositions into subvarieties. See Bittner [Bit04] for a careful discussion and presentation in terms of smooth varieties, and Looijenga [Loo02] for how localizations and completions of this ring give rise to motivic measures. The other definition originates in geometric model theory, as an instance of the Grothendieck ring of models of a first-order theory. Here, in line with the general aims of model theory, the objects of study are formulas of first order logic, and the subsets of an ambient model they define. The natural notion of morphism becomes a definable map, and in the Grothendieck ring of definable sets, it is natural to permit as relations all definable decompositions. Let us specialize to the theory of algebraically closed fields. Thanks to the existence of "elimination of quantifiers" from first order formulas, definable sets coincide with the loci of points satisfying a boolean combination of polynomial equalities in affine space, i.e. constructible sets. A morphism between constructible sets is a point-map whose graph is constructible. In this approach, varieties are seen as 'point-clouds' rather than as ringed spaces, and morphisms need not be continuous. See Krajíček and Scanlon [KS00] for a very readable exposition of this Grothendieck ring and some of its uses in logic.

There is a natural homomorphism from the algebraic geometer's Grothendieck ring of varieties, denoted $K_0(\mathsf{var}_k)$ in this paper, to the model theorist's, that we denote $K_0(\mathsf{constr}_k)$. This homomorphism is an isomorphism when k is algebraically closed of characteristic zero. This has been known in the model theory community for quite some time, and (as the author has learned

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after this work was completed) it follows from Prop. 3.8, Cor. 3.11 and Prop. 3.13 of Nicaise–Sebag [NS11]. It may still be useful to give a direct proof of this result, using only basic properties of separated, quasi-finite morphisms. We actually prove the slightly stronger result that the canonical comparison map from the Grothendieck semiring of varieties to the Grothendieck semiring of algebraically closed fields, is an isomorphism in characteristic zero. (Recall that a semiring is a ring-like structure without the requirement that additive inverses exist. In the Grothendieck semiring of varieties resp. algebraically closed fields, any element can be represented as a formal linear combination of objects with positive integer coefficients; hence, ultimately, as a single variety resp. constructible set. The Grothendieck semiring determines the corresponding Grothendieck ring, but not conversely.)

In positive characteristics, the situation is subtle. Conceptually, the reason for the difference is the absence in positive characteristic of generic smoothness. That makes it difficult to 'spread out' information given on the level of closed points, which is what is available in $K_0(\mathsf{constr}_k)$, to open subsets, and hence make a conclusion about $K_0(\mathsf{var}_k)$ using noetherian induction. We will prove that in positive characteristics, the canonical comparison map from the Grothendieck semiring $SK_0(\mathsf{var}_k)$ of varieties to those of constructible sets, $SK_0(\mathsf{constr}_k)$, is surjective but not injective. This leaves open the question whether $K_0(\mathsf{var}_k)$ and $K_0(\mathsf{constr}_k)$ are isomorphic in positive characteristics too. A resolution of this problem seems to require a better understanding of the canonical homomorphism $SK_0(\mathsf{var}_k) \to K_0(\mathsf{var}_k)$ in positive characteristics.

Let us give precise definitions. For an algebraically closed field k, let k-variety mean separated, reduced scheme of finite type over k. The Grothendieck semiring $SK_0(\mathsf{var}_k)$ is the commutative monoid (i.e. set with associative, commutative binary operation, with unit) generated by symbols $[X]$, one for each k-variety X, subject to the relations

- $[X] = [Y]$ if X and Y are isomorphic over k
- $[X] = [U] + [X - U]$ for any variety X with open subvariety U and closed complement $X - U$.

The product of k-varieties induces a commutative semiring structure on $SK_0(\mathsf{var}_k)$. The Grothendieck ring $K_0(\mathsf{var}_k)$ is defined analogously, based on the free abelian group generated by the symbols $[X]$.

A constructible subset of a scheme is one that can be written as a finite boolean combination of Zariski-closed subsets, considered as point-sets. Let constr_k be the category whose objects are pairs $\langle U, \mathbb{A}^n \rangle$ where U is a constructible subset of affine n-space $\mathbb{A}^n$ over k, and where a morphism $f : \langle U, \mathbb{A}^n \rangle \to \langle V, \mathbb{A}^m \rangle$ is a set-theoretic function $U \to V$ whose graph is a constructible subset of $\mathbb{A}^{n+m}$. The Grothendieck semiring $SK_0(\mathsf{constr}_k)$ of constructible sets is the commutative monoid generated by symbols $[\langle U, \mathbb{A}^n \rangle]$ corresponding to objects of constr_k, subject to the relations

- $[\langle U, \mathbb{A}^n \rangle] = [\langle V, \mathbb{A}^m \rangle]$ if $\langle U, \mathbb{A}^n \rangle$ and $\langle V, \mathbb{A}^m \rangle$ are isomorphic in constr_k
- $[\langle U, \mathbb{A}^n \rangle] = [\langle V, \mathbb{A}^n \rangle] + [\langle U - V, \mathbb{A}^n \rangle]$ whenever $V \subseteq U$.

For a scheme X, write $|X|$ for the set of points of its underlying topological space. Recall that there is a canonical (surjective) map $|X \times_k Y| \xrightarrow{p} |X| \times |Y|$. For constructible subsets $U \subseteq \mathbb{A}^n$,

$V \subseteq \mathbb{A}^m$, $p^{-1}(U \times V)$ is a constructible subset of $\mathbb{A}^{n+m}$. This turns $SK_0(\mathsf{constr}_k)$ into a semiring. The Grothendieck ring $K_0(\mathsf{constr}_k)$ is the commutative ring defined by the same generators and relations.

Given a finite decomposition of a variety X into pairwise disjoint affine constructible sets $C_i \subseteq \mathbb{A}^{d_i}$, $i \in I$, define

$$\alpha_S(X, I) = \sum_{i \in I} \left[\left\langle C_i,\ \mathbb{A}^{d_i} \right\rangle \right] \tag{1.1}$$

as an element of $SK_0(\mathsf{constr}_k)$.

Proposition 1.1. *$\alpha_S(X, I)$ yields a well-defined homomorphism $SK_0(\mathsf{var}_k) \xrightarrow{\alpha_S} SK_0(\mathsf{constr}_k)$.*

Indeed, such a decomposition exists for every variety X: choose an affine atlas $\{U_i\}_{1 \leqslant i \leqslant n}$ and for $1 \leqslant i \leqslant n$ let

$$C_i = U_i - \Big(\bigcup_{1 \leqslant j < i} U_j \Big)$$

Then $X = \bigsqcup_{1 \leqslant i \leqslant n} C_i$ with embeddings $C_i \subseteq \mathbb{A}^{\dim(X)}$.

The class of $\alpha_S(X, -)$ is independent of the decomposition chosen. Indeed, if C_i, $i \in I$ and D_j, $j \in J$ are two such decompositions of X then $\{C_i \cap D_j \mid (i,j) \in I \times J\}$ refines both of them, and it easily follows that

$$\alpha_S(X, I) = \alpha_S(X, I \times J) = \alpha_S(X, J)$$

in $SK_0(\mathsf{constr}_k)$. That α_S respects the relation $[X] = [U] + [X - U]$ in $SK_0(\mathsf{var}_k)$ follows similarly, by intersecting a decomposition of X with U resp. $X - U$. $\square$

There exists a functorial homomorphism ϕ from any Grothendieck semiring to the corresponding Grothendieck ring, yielding a commutative diagram

$$\begin{tikzcd} SK_0(\mathsf{var}_k) \arrow[r, "\alpha_S"] \arrow[d, "\phi_{\mathsf{var}}"] & SK_0(\mathsf{constr}_k) \arrow[d, "\phi_{\mathsf{constr}}"] \\ K_0(\mathsf{var}_k) \arrow[r, "\alpha"] & K_0(\mathsf{constr}_k) \end{tikzcd}$$

Since any affine constructible set can be decomposed as a finite disjoint union of locally closed subsets that are (the underlying point-sets of) varieties, both α_S and α are surjective. Working in characteristic zero, recent preprints of Karzhemanov [Kar14] and Borisov [Bor15] give explicit varieties X, Y whose classes are different in $SK_0(\mathsf{var}_k)$ but equal in $K_0(\mathsf{var}_k)$, while Liu–Sebag [LS10] give sufficient conditions on X, Y such that if $[X] = [Y]$ in $K_0(\mathsf{var}_k)$ then $[X] = [Y]$ in $SK_0(\mathsf{var}_k)$ (that is, X and Y are "cut-and-paste equivalent" or "piecewise isomorphic").

The main results of this paper are

Theorem 2.9. If $\mathrm{char}(k) = 0$ then α_S is an isomorphism.

By functoriality, this implies that so is α. The proof occupies the next section.

Theorem 3.2. If $\mathrm{char}(k) > 0$ then α_S is not injective.

The proof of this uses the simple

Lemma 1.2. *Suppose the morphism $V \xrightarrow{f} W$ of varieties induces a bijection $V(k) \to W(k)$ on k-points. Then $\alpha_S[V] = \alpha_S[W]$ in $SK_0(\mathsf{constr}_k)$.*

Indeed, let $\{A_i\}$ be an affine atlas of V and $\{B_j\}$ an affine atlas of W. Let $\{C_i\}$ be a decomposition of V into constructible subsets that refines both $\{A_i\}$ and $\{f^{-1}(B_j)\}$. Since k-points are dense, the collection $\{f(C_i)\}$ must be a decomposition of W into constructible subsets. Since each C_i resp. $f(C_i)$ is affine constructible and f induces a bijection between their k-points, $\alpha_S[V] = \alpha_S[W]$ follows. □

In section 3, we give an example of a morphism of varieties $V \xrightarrow{f} W$ in any positive characteristic such that $[V] \neq [W]$ in $SK_0(\mathsf{var}_k)$ but f induces a bijection on k-points. If it were true that $[V] \neq [W]$ in $K_0(\mathsf{var}_k)$ as well, then one could conclude that α is not injective either. However, even though V and W can be taken to be curves, the positive characteristic assumption prevents one from applying the results of Liu and Sebag. It seems natural to conjecture that α is not injective in positive characteristics. To prove that, one should probably combine the type of counterexamples constructed here, involving Frobenius twists, with a study of cohomological invariants in positive characteristics that descend to the Grothendieck ring of varieties.

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2. The separable case

In this section, we continue to assume that k is algebraically closed and work in the category of varieties over k.

Lemma 2.1. *Let $V \xrightarrow{f} W$ be a separable morphism that induces a bijection $V(k) \to W(k)$ on k-points. Then there exist stratifications of V and W into locally closed subvarieties*

$$V = \bigsqcup_{i=1}^{n} V_i \quad \text{resp.} \quad W = \bigsqcup_{i=1}^{n} W_i$$

such that f restricts to an isomorphism $V_i \to W_i$ for $i = 1, 2, \ldots, n$. Hence $[V] = [W]$ in $SK_0(\mathsf{var}_k)$.

Proof. Let U be a top-dimensional irreducible component of V and let Z be the Zariski closure of $f(U)$ in W. Then Z is irreducible too. Since the fibers of f have dimension $\dim(U) - \dim(Z)$ generically over Z and f is bijective, $\dim(U) = \dim(Z)$. Since $U \xrightarrow{f} Z$ is dominating, $k(U)$ is a finite extension of $k(Z)$. Let n be the separable degree of $k(U)$ over $k(Z)$; since the fibers of f have cardinality n generically over Z, $n = 1$. By the assumption that f is separable, $k(U) = k(Z)$. Thus f restricts to an isomorphism on an open subset of U. Noetherian induction now gives the conclusion. □

There are many variants of this basic lemma. For example, additional hypotheses yield the conclusion that f itself is an isomorphism:

Proposition 2.2. *Let V, W be irreducible varieties and $V \xrightarrow{f} W$ a separable morphism that induces a bijection on k-points. Assume W is normal. Then f is an isomorphism.*

Cf. Milne [Mil14] Theorem 8.40 and the surrounding discussion about the necessity of the hypotheses.

Alternatively, in the presence of smoothness, one can conclude that f is étale and, eventually, an open immersion:

Proposition 2.3. *Assume* $\mathrm{char}(k) = 0$ *and let $V \xrightarrow{f} W$ be a morphism of varieties that induces a bijection on k-points and is smooth at some point $x \in V$. Then on an open neighborhood U of x, $f|_U$ is an isomorphism.*

Indeed, by the structure theorem for smooth morphisms, there exists an open neighborhood U of x such that $f(U)$ is an open neighborhood of $f(x)$, and $f|_U$ can be factored as

$$U \xrightarrow{q} f(U) \times_k \mathbb{A}^d \xrightarrow{\mathrm{pr}} f(U)$$

where q is étale, pr is projection on the first factor, and $d = \dim(U) - \dim\big(f(U)\big)$. Since f is bijective on k-points, $d = 0$; so $f|_U$ is étale. By EGA IV_4, Théorème 17.9.1, an étale morphism is an open immersion if and only if it is radicial, that is, injective on underlying points with purely inseparable residue field extensions. Because of the characteristic zero assumption, this last condition is tautologous (the residue field extensions being the identity).

Corollary 2.4. *Assume* $\mathrm{char}(k) = 0$ *and $V \xrightarrow{f} W$ is a morphism of varieties that induces a bijection on k-points. Then $[V] = [W]$ in $SK_0(\mathsf{var}_k)$.*

This follows from Lemma 2.1 since the $\mathrm{char}(k) = 0$ assumption guarantees separability; or, from Prop. 2.2 by noetherian induction on the smooth locus of the base; or, from Prop. 2.3 by generic smoothness for morphisms and noetherian induction.

Remark 2.5. It is well-known that in the absence of additional assumptions, one cannot conclude in the situation of the corollary that f itself is an isomorphism. The canonical example of a finite, bijective morphism that is not an isomorphism is the regular map $\mathbb{A}^1 \xrightarrow{f} C$ given by $t \to (t^2, t^3)$ where C is the cuspidal affine cubic defined by $X^3 - Y^2 = 0$. Though not an isomorphism, f is a birational equivalence inducing a bijection $\mathbb{A}^1(k) \to C(k)$ for all fields k. This is running 'Example O' in Mumford [Mum99], Chapters I and II.

For the rest of this section, we keep the assumption that $\mathrm{char}(k) = 0$. The first theorem of the introduction follows by easy (though aggravating) bookkeeping.

Lemma 2.6. *Suppose $\langle U, \mathbb{A}^u\rangle$ and $\langle V, \mathbb{A}^v\rangle$ are isomorphic in constr_k. Let $\{U_i \mid i \in I\}$ resp. $\{V_i \mid i \in J\}$ be arbitrary decompositions of U (resp. V) into finitely many locally closed subvarieties of $\mathbb{A}^u$ (resp. $\mathbb{A}^v$). Then*

$$\sum_{i\in I} [U_i] = \sum_{i\in J} [V_i]$$

in $SK_0(\mathsf{var}_k)$.

The assumption that $\langle U, \mathbb{A}^u\rangle$ and $\langle V, \mathbb{A}^v\rangle$ are isomorphic in constr_k means that there exists a constructible subset $\Gamma \subset \mathbb{A}^{u+v}$ such that the projection $\mathbb{A}^{u+v} \xrightarrow{\mathrm{pr}_1} \mathbb{A}^u$ induces a bijection between the k-points of Γ and U, and similarly for $\mathbb{A}^{u+v} \xrightarrow{\mathrm{pr}_2} \mathbb{A}^v$ between $\Gamma(k)$ and $V(k)$. $\mathrm{pr}_1^{-1}(U_i)$, $i \in I$ and $\mathrm{pr}_2^{-1}(V_i)$, $i \in J$, are decompositions of Γ into constructible subsets. Find a finite decomposition $\{W_q \mid q \in Q\}$ of Γ into locally closed subvarieties of $\mathbb{A}^{u+v}$ that refines both. Take now the coproduct of the W_q or, more concretely, choose distinct points $t_q \in \mathbb{A}^1$ for $q \in Q$, and let $\Gamma^+ = \coprod_{q\in Q} W_q \times \{t_q\} \subset \mathbb{A}^{u+v+1}$. Consider the map $p_1 : \Gamma^+ \hookrightarrow \mathbb{A}^{u+v+1} \xrightarrow{\mathrm{pr}} \mathbb{A}^u$. Apply Cor. 2.4 to the restriction of p_1 to $\Gamma^+ \cap p_1^{-1}(U_i) \to U_i$, for each i, and sum over $i \in I$ to obtain

$$\sum_{i\in I} [U_i] = \sum_{q\in Q} [W_q]$$

in $SK_0(\mathsf{var}_k)$. The same argument for the other projection establishes

$$\sum_{q\in Q} [W_q] = \sum_{i\in J} [V_i]$$

as desired. □

The following corollary is not necessary for the main result, but worth pointing out.

Corollary 2.7. *Suppose $\langle U, \mathbb{A}^u\rangle$ and $\langle V, \mathbb{A}^v\rangle$ are isomorphic in constr_k. Then there exist decompositions $U = \coprod_{i=1}^n W_i^{(0)}$ and $V = \coprod_{i=1}^n W_i^{(1)}$ into locally closed subvarieties such that $W_i^{(0)}$ and $W_i^{(1)}$ are isomorphic as k-varieties, for each $i = 1, 2, \ldots, n$.*

Indeed, whenever $\sum_{i\in I} [U_i] = \sum_{i\in J} [V_i]$ in $SK_0(\mathsf{var}_k)$ then the collections $\{U_i\}$ and $\{V_i\}$ are "scissors equivalent". That is, there exist a finite set of varieties W_p, $p \in P$, and maps $f : P \to I$, $g : P \to J$ so that for each $i \in I$, U_i can be decomposed into subvarieties isomorphic to the collection $\{W_p \mid p \in f^{-1}(i)\}$ and for each $j \in J$, V_j can be decomposed into subvarieties isomorphic to the collection $\{W_p \mid p \in g^{-1}(j)\}$. To see this, use induction on the number of relations needed to change $\sum_{i\in I} [U_i]$ into $\sum_{i\in J} [V_i]$ in $SK_0(\mathsf{var}_k)$ and use the fact that any two stratifications of a variety into locally closed subvarieties have a common refinement.

Proposition 2.8. *Suppose $\sum_{i\in I} [\langle U_i, \mathbb{A}^{u_i}\rangle] = \sum_{i\in J} [\langle V_i, \mathbb{A}^{v_i}\rangle]$ in $SK_0(\mathsf{constr}_k)$. For each i, let U_i be decomposed into finitely many locally closed subvarieties $\{U_{i,j}^{(0)} \mid j \in I_i\}$ of $\mathbb{A}^{u_i}$, and similarly for the V_i into $\{V_{i,j}^{(0)} \mid j \in J_i\}$. Then*

$$\sum_{i\in I}\sum_{j\in I_i} [U_{i,j}^0] = \sum_{i\in J}\sum_{j\in J_i} [V_{i,j}^{(0)}]$$

in $SK_0(\mathsf{var}_k)$.

To see this, use induction on the number of relations needed to change $\sum_{i\in I} [\langle U_i, \mathbb{A}^{u_i}\rangle]$ into $\sum_{i\in J} [\langle V_i, \mathbb{A}^{v_i}\rangle]$. If the formal expressions are the same (up to rearrangement), the claim follows from Lemma 2.6 applied to the identity. If the relation being used is of the form $[\langle U, \mathbb{A}^{u}\rangle] = [\langle V, \mathbb{A}^{v}\rangle]$ for isomorphic constructible sets, apply Lemma 2.6. If the relation being used is of the form $[\langle U, \mathbb{A}^{n}\rangle] = [\langle V, \mathbb{A}^{n}\rangle] + [\langle U - V, \mathbb{A}^{n}\rangle]$, Lemma 2.6 applies again, since the union of decompositions of V and $U - V$ into locally closed subvarieties of $\mathbb{A}^n$ is such a decomposition of U.

Theorem 2.9. *If* $\mathrm{char}(k) = 0$ *then* $SK_0(\mathsf{var}_k) \xrightarrow{\alpha_S} SK_0(\mathsf{constr}_k)$ *is an isomorphism.*

Proof. One only needs to prove that α_S is injective. Note that we defined variety to mean separated, reduced scheme of finite type over k; in particular, we work freely with non-connected varieties. Thus any element of $SK_0(\mathsf{var}_k)$ can be represented by the class $[X]$ of a variety X. Suppose

$$\alpha_S[X] = \sum_{i\in I} [\langle U_i, \mathbb{A}^{u_i}\rangle] = \sum_{i\in J} [\langle V_i, \mathbb{A}^{v_i}\rangle] = \alpha_S[Y]$$

with $\{U_i \mid i \in I\}$ and $\{V_i \mid i \in J\}$ constructible decompositions of X resp. Y. The conclusion of Prop. 2.8 now implies that $[X] = [Y]$ in $SK_0(\mathsf{var}_k)$. □

3. POSITIVE CHARACTERISTIC

Lemma 2.1 hints that the failure of α_S to be an isomorphism will be due to inseparable morphisms, such as the Frobenius. Since we work over algebraically closed fields, we need the relative Frobenius.

Let p be a prime. Recall that the absolute Frobenius Fr_p is a functorial self-map $X \xrightarrow{\mathrm{Fr}_p(X)} X$ in $\mathsf{Schemes}/\mathbb{F}_p$ that acts as the identity on the underlying topological space of X and as p^{th} power on its structure sheaf. Stated differently, Fr_p acts as the Frobenius endomorphism $x \mapsto x^p$ for $x \in A$ on an $\mathbb{F}_p$-algebra A, and this extends from affine to all $\mathbb{F}_p$-schemes via patching.

Let $f : X \to S$ be a morphism in $\mathsf{Schemes}/\mathbb{F}_p$. Apply Fr_p to it; let $X^{(p)}$ be its pullback along $\mathrm{Fr}_p(S)$ and let the *relative Frobenius* Fr_S be the canonical map $X \to X^{(p)}$ into the pullback:

$$\begin{tikzcd}
X \arrow[dr, "\mathrm{Fr}_S"'] \arrow[drr, bend left, "\mathrm{Fr}_p(X)"] \arrow[ddr, bend right, "f"'] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& S \arrow[r, "\mathrm{Fr}_p(S)"] & S
\end{tikzcd}$$

$X \mapsto X^{(p)}$ gives a functor and $X \xrightarrow{\mathrm{Fr}_S} X^{(p)}$ a natural transformation on $\mathsf{Schemes}/S$.

Example 3.1. Let k be an $\mathbb{F}_p$-algebra, $\bar{x} = \{x_i \mid i \in I\}$ a set of variables, I an ideal in $k[\bar{x}]$. For

$$f(\bar{x}) = \sum a_{i_1,i_2,\ldots,i_m} x_{i_1}^{n_1} x_{i_2}^{n_2} \ldots x_{i_m}^{n_m} \in k[\bar{x}]$$

write

$$f(\bar{x})^{(p)} = \sum a^p_{i_1,i_2,\ldots,i_m} x^{n_1}_{i_1} x^{n_2}_{i_2} \ldots x^{n_m}_{i_m} \in k[\bar{x}] \ .$$

Let $S = \operatorname{spec}(k)$, $X = \operatorname{spec} k[\bar{x}]/I$ and $I^{(p)} = \{f(\bar{x})^{(p)} \mid f(\bar{x}) \in I\}$. Then $X^{(p)} = \operatorname{spec} k[\bar{x}]/I^{(p)}$. The relative Frobenius $\mathrm{Fr}_S : X \to X^{(p)}$ is generated by $x_i \mapsto x_i^p$.

When k is a field of characteristic p, we will write Fr_k for $\mathrm{Fr}_{\mathrm{spec}(k)}$. (Note that the absolute Frobenius Fr_p is identical with the relative Frobenius Fr_k over $k = \mathbb{F}_p$; our notation is meant to reflect this. The notation $\mathrm{Fr}_{X/S}$ is common for what we denote $\mathrm{Fr}_S : X \to X^{(p)}$ here.)

For any scheme X over a field k of characteristic p, $X(k) \xrightarrow{\mathrm{Fr}_k} X^{(p)}(k)$ is injective, since $X(k) \xrightarrow{\mathrm{Fr}_p} X(k)$ is so. If k is perfect, then $X(k) \xrightarrow{\mathrm{Fr}_k} X^{(p)}(k)$ is surjective, since $X(k) \xrightarrow{\mathrm{Fr}_p} X(k)$ is so and $X^{(p)}(k) \to X(k)$, being a pullback of Fr_p, is injective. From now on, assume k algebraically closed. Then $X(k)$ and $X^{(p)}(k)$ will biject through Fr_k, implying $\alpha_S[X] = \alpha_S[X^{(p)}]$ by Lemma 1.2.

Via the j-invariant, k-isomorphism classes of elliptic curves biject with k. Let $a \in k$ be such that $a^p \neq a$ (i.e. a does not belong to the prime field) and let E be an elliptic curve with j-invariant a. Since the j-invariant is a rational function of the coefficients of the Weierstrass form of E (see e.g. Silverman [Sil09] for explicit formulas), the Frobenius twist $E^{(p)}$ has j-invariant a^p. That is, E and $E^{(p)}$ are not isomorphic over k.

It is easy to see that $[E] \neq [E^{(p)}]$ in $SK_0(\mathsf{var}_k)$. Indeed (working now in arbitrary characteristic) if C_1 and C_2 are smooth, irreducible, complete curves such that $[C_1] = [C_2]$ in $SK_0(\mathsf{var}_k)$ then there must exist decompositions $C_1 = C_1^0 \bigcup \{p_1, p_2, \ldots, p_n\}$ and $C_2 = C_2^0 \bigcup \{q_1, q_2, \ldots, q_m\}$ where p_i, q_i are points, such that C_1^0 and C_2^0 are isomorphic (since these are the only possible forms of relations that apply to irreducible curves in $SK_0(\mathsf{var}_k)$.) But then C_1 and C_2 are isomorphic too, being the completions of C_1^0 resp. C_2^0 (and hence $n = m$).

To sum up:

Theorem 3.2. *If k is algebraically closed of positive characteristic, then there exist curves V, W such that $[V] \neq [W]$ in $SK_0(\mathsf{var}_k)$ but $\alpha_S[V] = \alpha_S[W]$ in $SK_0(\mathsf{constr}_k)$.*

Note that Proposition 6 of Liu–Sebag [LS10] states that for varieties V, W of dimension 1, if $[V] = [W]$ in $K_0(\mathsf{var}_k)$ where k is algebraically closed *of characteristic zero* then (in the terminology of Liu and Sebag) "V and W are piecewise isomorphic", that is, $[V] = [W]$ in $SK_0(\mathsf{var}_k)$. The removal of the characteristic zero assumption there would also extend Theorem 3.2 from the Grothendieck semiring to the usual Grothendieck ring.

References

[Bit04] Franziska Bittner, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. **140** (2004), no. 4, 1011–1032.

[Bor15] Lev A. Borisov, *The class of the affine line is a zero divisor in the Grothendieck ring*, arXiv:1412.6194v3 [math.AG], 2015.

[Kar14] Ilya Karzhemanov, *On the cut-and-paste property of algebraic varieties*, arXiv:1411.6084v3 [math.AG], 2014.

[KS00] Jan Krajíček and Thomas Scanlon, *Combinatorics with definable sets: Euler characteristics and Grothendieck rings*, Bull. Symbolic Logic (2000), no. 3, 311–330.

[Loo02] Eduard Looijenga, *Motivic measures*, Astérisque (2002), no. 276, 267–297.

[LS10] Qing Liu and Julien Sebag, *The Grothendieck ring of varieties and piecewise isomorphisms*, Mathematische Zeitschrift **265** (2010), 321–342.

[Mil14] James Milne, *Algebraic Geometry*, Lecture notes ver. 6.00, 2014, available at http://www.jmilne.org/.

[Mum99] David Mumford, *The Red Book of Varieties and Schemes*, Springer-Verlag, 1999.

[NS11] Johannes Nicaise and Julien Sebag, *The Grothendieck ring of varieties*, Motivic integration and its interactions with model theory and non-Archimedean geometry, vol. I, London Math. Soc. Lecture Note Ser., no. 383, Cambridge University Press, 2011, pp. 145–188.

[Sil09] Joseph Silverman, *The Arithmetic of Elliptic Curves*, second ed., Springer-Verlag, 2009.

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