The Grothendieck (semi)ring of algebraically closed fields

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Nov 3, 2015
First order logic: language contains relation, constant and function symbols; only allows $\forall$ and $\exists$ quantifiers ranging over the domain of interpretation

‘Theory’ is a set of first order axioms in a given language

Can axiomatize most algebraic structures (e.g: ring, field, difference field; algebraically closed field, real closed field; category, groupoid; metric space) this way

No obvious first order axiomatization for: noetherian ring; topological space; manifold; variety; ringed space; scheme; complete metric space
Strong set-theoretical flavor. Motivating question: given a theory $T$ and infinite cardinal $\kappa$, what is the cardinality of the set of isomorphism classes of models of $T$ of size $\kappa$?

Take $T$ to be the theory of algebraically closed fields of a specific characteristic, $\kappa$ an uncountable cardinal. Any two models of $T$ of size $\kappa$ are isomorphic (since they must have the same transcendence degree, namely $\kappa$, over the prime field). This is a rare phenomenon!

Łós’s conjecture: Let $T$ contain countably many axioms. Suppose that for at least one uncountable $\kappa$, $T$ has a unique isomorphism class of models of cardinality $\kappa$. Then, for every uncountable $\kappa$, $T$ has a unique isomorphism class of models of cardinality $\kappa$. 

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Morley (1966) proves Łós’s conjecture. Major contribution: identifies a class of theories \( T \) whose models admit a structure theory of ‘transcendence degree’ similar to those of algebraically closed fields. (These theories are called \( \omega \)-stable.)

Vast technical elaboration of machinery of stability by Shelah, Baldwin, Lachlan, and others. Solution of the ‘spectrum problem’: how many isomorphism classes of models can a theory have in uncountable cardinalities.

Interest shifts to understanding the fine structure of specific models of specific types of theories.
model theory, ca. 1990 – 2000
A model-theoretic geometry consists of a set $X$ ("points") and for each $n \in \mathbb{N}$, a set of subsets of $X^n$, denoted $\mathcal{B}_n$ ("definable subsets of $X^n$") such that

- $\mathcal{B}_n$ is closed under boolean operations in $X^n$
- if $U \in \mathcal{B}_n$ and $V \in \mathcal{B}_m$ then $U \times V \in \mathcal{B}_{n+m}$
- if $U \in \mathcal{B}_n$ then $pr(U) \in \mathcal{B}_m$ for any projection $pr : X^n \rightarrow X^m$
- diagonals belong to $\mathcal{B}_n$; singletons belong to $\mathcal{B}_1$.

See van den Dries: *Tame geometry and o-minimal structures* for a minimal set of axioms.
Let $X, B_n, n \in \mathbb{N}$ be a model-theoretic geometry. Introduce the category

$$\text{Def}(X) \quad \begin{cases} \text{objects} & = \text{definable sets (i.e. elements of } B_n) \\ \text{morphisms} & = \text{definable functions} \end{cases}$$

i.e. a morphism from a definable $U \subseteq X^n$ to a definable $V \subseteq X^m$ is a function $f : U \to V$ whose graph belongs to $B_{n+m}$. 

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Let $\mathcal{L}$ be a first-order signature (set of constant, function and relation symbols), and let the set $X$ be equipped with interpretations of these symbols. For $U \subseteq X^n$, let

$$U \in \mathcal{B}_n \quad \text{iff} \quad U = \{ \mathbf{x} \in X^n \mid X \models \phi(\mathbf{x}) \}$$

for some first-order formula $\phi$ in the signature $\mathcal{L}$ (allowing parameters form $X$), with free variables from among the $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$. 

The Grothendieck (semi)ring of algebraically closed fields
motivating example: $\text{SemiLin}_\mathbb{R}$

underlying set: $\mathbb{R}$; language: $+$, scalar multiplication, $< =$

- can define half-spaces $\{x \mid \langle a, x \rangle < b\}$, affine subspaces and their finite boolean combinations; and via first order formulas, only these
- objects of $\text{SemiLin}_\mathbb{R}$ are finite boolean combinations of polytopes (possibly unbounded)
- morphisms of $\text{SemiLin}_\mathbb{R}$ are “piecewise linear” functions (i.e. set-functions whose graph belongs to $\text{SemiLin}_\mathbb{R}$; need not be continuous!)
- the $n$-simplex $\Delta_n$ and $[0, 1]^n$ are isomorphic in $\text{SemiLin}_\mathbb{R}$ (fun!)
motivating example: $FO(k)$

Let $k$ be a field and let $FO(k)$ be the geometry of first order definable sets over $k$, in the language of $+ \cdot =$

Best understood when $k$ is a local field, or an algebraically closed field (or “close” to being algebraically closed: pseudo-finite, pseudo-algebraically closed etc . . .)
Consider $FO(\mathbb{R})$. Relation $x < y$ is definable as $x \neq y \land \exists z (x + z^2 = y)$. So $FO(\mathbb{R})$ contains all semi-algebraic sets and in fact, coincides with semi-algebraic sets.

- subset of $\mathbb{R}^n$ is semi-algebraic if it can be written as a finite boolean combination of sets of the form $\{x \mid p(x) > 0\}$, where $p(x)$ is a polynomial.
- projection of semi-algebraic set is semi-algebraic (Seidenberg), equivalently: the subset of $\mathbb{R}^n$ defined by any first order formula in the above language is semi-algebraic (Tarski).
- objects of $FO(\mathbb{R}) = SemiAlg_{\mathbb{R}}$ are semi-algebraic subsets of $\mathbb{R}^n$.
- morphisms are set-functions with semi-algebraic graph (need not be continuous!)
Let $k$ be an algebraically closed field. $FO(k)$ will coincide with $\text{Constr}_k$, the category whose

- objects are constructible subsets of $k^n$ (closed under projection by Chevalley’s theorem; Tarski also proves that any subset of $k^n$ definable via a first order formula in the above language, is constructible)

- morphisms are set-functions with constructible graph (need not be continuous!)
main problems concerning the category of definable sets

- Does it have quotients of equivalence relations?
- Find a notion of ‘dimension’ for objects

- Find a notion of ‘size’ for objects

- How do these invariants vary in families?
- Is there a field (perhaps even algebraically closed) among the objects of Def(X)?
- What are (abelian) group objects in Def(X)?
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main problems concerning the category of definable sets

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dimension will be a homomorphism from the Grothendieck semiring to some semi-lattice
- Find a notion of ‘size’ for objects
  (finitely additive) measure will be a homomorphism from the Grothendieck group to some abelian group; Euler characteristic will be homomorphism from the Grothendieck ring to some ring
- How do these invariants vary in families?
- Is there a field (perhaps even algebraically closed) among the objects of Def(X),en
- What are (abelian) group objects in Def(X)?
Identify combinatorial conditions on first-order theories $T$ that ensure ‘nice solutions’ to the main problems.

http://www.forkinganddividing.com
Proposition $\text{Def}(X)$

- has terminal object and pullbacks (so finite limits); they are computed as in Set
- has finite coproducts
- is distributive: the canonical maps

$$
\emptyset \rightarrow X \times \emptyset
$$

$$
X \times Y \sqcup X \times Z \rightarrow X \times (Y \sqcup Z)
$$

are isomorphisms

- is boolean (subobject posets are boolean algebras; every subobject is a coproduct summand)
Grothendieck (semi)ring of a (small) distributive category $\mathcal{C}$

$SK(\mathcal{C})$ is the semiring whose elements are isomorphism classes $[X]$ of objects $X$, with $[X] \cdot [Y] := [X \times Y]$ and $[X] + [Y] := [X \sqcup Y]$.

$K(\mathcal{C})$ is the abelian group generated by isomorphism classes $[X]$ of objects $X$, with the relations $[X \sqcup Y] = [X] + [Y]$. Multiplication is induced by $[X] \cdot [Y] = [X \times Y]$.

• **Semiring** is a ‘ring without additive inverses’.

• There are adjoint functors

\[
\begin{align*}
\text{Ring} & \overset{\text{inc}}{\leftrightarrow} \text{SemiRing} \\
& \overset{\text{groth}}{\leftrightarrow} \text{SemiRing}
\end{align*}
\]

and $K(\mathcal{C}) = groth(SK(\mathcal{C}))$.

• Schanuel (1990) calls $SK(\mathcal{C})$ the “Burnside rig of $\mathcal{C}$” in his pioneering article *Negative sets have Euler characteristic and dimension*. 
**Theorem** (Lojasiewicz; Hironaka, ca. 1960) The inclusion of categories $\text{SemiLin}_\mathbb{R} \hookrightarrow \text{SemiAlg}_\mathbb{R}$ induces isomorphisms

$$SK(\text{SemiLin}_\mathbb{R}) \xrightarrow{\cong} SK(\text{SemiAlg}_\mathbb{R})$$

**Theorem** (Schanuel 1990) $SK(\text{SemiLin})$ is a finitely presentable semiring, isomorphic to

$$\mathbb{N}[X]/(X = 2X + 1).$$

**Theorem** (Schanuel 1990; o-minimal: van den Dries, 1998) There is a monomorphism

$$SK(\text{SemiAlg}) \xrightarrow{\text{dim} \times \text{eu}} \{\mathbb{N} \cup -\infty\}_{(+, \text{max})} \times \mathbb{Z}$$

where $\text{dim}$ is topological dimension and $\text{eu}$ is the combinatorial Euler characteristic (to be defined momentarily).
Schanuel’s presentation

Commutative diagram

\[
\begin{align*}
\mathbb{N}[X]/(X = 2X + 1) & \xrightarrow{\text{deg} \times \{(f(X) \mapsto f(-1)\}} \mathbb{N}_{-\infty} \times \mathbb{Z} \\
X \mapsto (0,1) & \quad \quad \quad \dim \times \text{eu} \\
SK(SemiAlg) &
\end{align*}
\]

Degree-wise induction shows top arrow injective; left arrow surjective, hence isomorphism.
Commutative diagram

\[
SK(SemiAlg) \xrightarrow{\dim \times eu} \{\mathbb{N} \cup -\infty\} \times \mathbb{Z} \\
\downarrow groth \hspace{2cm} \downarrow groth \\
K(SemiAlg) \xrightarrow{eu} \mathbb{Z}
\]

groth preserves products. It need not preserve monos, but an easy argument shows that in the present case, the bottom arrow is an isomorphism.
Let $X$ be semi-algebraic and $(V, S)$ an open-cell complex such that $X$ is semi-algebraically homeomorphic to $|S|$.

**Definition** $\text{eu}(X) = \sum_{U \in S} (-1)^{\dim(U)}$

**Theorem** $\text{eu}(X)$ is independent of the open-cell decomposition chosen.

The proof needs that any two semi-algebraic open-cell decompositions have a common semi-algebraic refinement.
Let $\mathbb{F}$ be any field. Let $H^*(-; \mathbb{F})$ denote sheaf (or equivalently, singular) cohomology and let $H^*_c(-; \mathbb{F})$ denote cohomology with compact support. Let $X$ be a semi-algebraic set.

If $X$ is locally compact,

$$\text{eu}(X) = \chi_c(X) = \sum_{i=0}^{\dim(X)} (-1)^i \dim_{\mathbb{F}} H^i_c(X; \mathbb{F}).$$

Follows from $H^*_c$ long exact sequence of $U \subset X$ where $U$ is open, $X$ Hausdorff, locally compact; cell decomposition of $X$ and induction.
embarrassing!

- Is \( \text{eu}(X) = \chi_c(X) \) for all semi-algebraic \( X \), not just locally compact ones? Is there a cohomological interpretation of \( \text{eu}(X) \) valid for all \( X \)?

- \( \chi_c \) is a proper (topological) homotopy invariant. Is that true for \( \text{eu}(X) \) as well?

**Theorem** (TB, 2011) If \( X, Y \) are semi-algebraic (or more generally, o-minimal, belonging to possibly two distinct o-minimal structures) and topologically homeomorphic then \( \text{eu}(X) = \text{eu}(Y) \).

Proof reduces to locally compact case with the help of an intrinsically defined stratification of o-minimal sets.

**Remark** Already two polyhedra can be topologically homeomorphic but not semi-algebraically so (Milnor, counterexample to the polyhedral Hauptvermutung, 1961).
Recall that $FO(\mathbb{Q}_p)$ is the geometry of subsets of $(\mathbb{Q}_p)^n$ that are first order definable in the language $+ \cdot =$. 

**Theorem** (Clucker–Haskell 2000) The Grothendieck ring of $FO(\mathbb{Q}_p)$ is trivial.

When $p \neq 2$

$$\mathbb{Z}_p = \{ x \in \mathbb{Q}_p | \exists y \in \mathbb{Q}_p (y^2 = 1 + px^2) \}$$

When $p = 2$

$$\mathbb{Z}_2 = \{ x \in \mathbb{Q}_2 | \exists y \in \mathbb{Q}_2 (y^3 = 1 + 2x^3) \}$$

So $\mathbb{Z}_p$ is an object of $FO(\mathbb{Q}_p)$. 
\( \mathbb{Z}_p \) is an object of \( FO(\mathbb{Q}_p) \). Clucker and Haskell then give an explicit bijection between \( \mathbb{Z}_p - \{0\} \) and \( \mathbb{Z}_p \) in \( FO(\mathbb{Q}_p) \).

In any distributive category \( C \), if for some object \( Z \) the objects \( Z - \{pt\} \) and \( Z \) are isomorphic, then \([pt] = [\emptyset]\) in \( K(C) \), so the Grothendieck ring \( K(C) \) is trivial.
For any field $k$, let $SK(Var_k)$ be the semiring with generators the varieties over $k$ and relations

$$[X] = [Y] \quad \text{if } X \text{ and } Y \text{ are isomorphic over } k$$

$$[X] = [X - U] + [U]$$

for every open subvariety $U$ of $X$ with complement $X - U$.

The product of $[X]$ and $[Y]$ is $[X \otimes_k Y]$.

$K(Var_k)$ is the ring generated by the same generators and relations.
Let $k$ be algebraically closed. There’s a natural homomorphism

$$SK(Var_k) \xrightarrow{\alpha_S} SK(Constr_k)$$

**Theorem** (TB 2013) $\alpha_S$ is an isomorphism when $\text{char}(k) = 0$. It is surjective but not injective when $\text{char}(k) > 0$.

**Corollary** (folk; Sebag-Nicaise 2011) The model-theorist’s Grothendieck ring of the field $k$ and Grothendieck’s Grothendieck ring of varieties over $k$, are isomorphic for $k$ algebraically closed of characteristic 0.
the comparison homomorphism

$$SK(\text{Var}_k) \xrightarrow{\alpha_S} SK(\text{Constr}_k)$$

Given variety $X$, choose a decomposition (as point set) $X = \bigcup_{i=1}^n C_i$ into pairwise disjoint affine constructible sets and send $[X]$ to $\sum_{i=1}^n [C_i] \in SK(\text{Constr}_k)$.

- such decompositions always exist; e.g. choose an affine atlas $\{U_i \mid i = 1, 2, 3, \ldots, n\}$ and set
  $$C_i := U_i - \left(\sum_{j=1}^{i-1} U_j\right)$$

- $\sum_{i=1}^n [C_i]$, as element of $SK(\text{Constr}_k)$, is independent of decomposition chosen

- $\alpha_S$ is compatible with $+$ and $\times$.  

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Proposition

\[ SK(\text{Var}_k) \xrightarrow{\alpha_S} SK(\text{Constr}_k) \]

is onto (in every characteristic).

Use the fact that any constructible subset of \( \mathbb{A}^n_k \) can be stratified as a disjoint union of locally closed subvarieties of \( \mathbb{A}^n_k \).
**Key fact** (cf. Zariski’s main theorem) Let $V, W$ be irreducible varieties and $V \xrightarrow{f} W$ a separable morphism that induces a bijection on $k$-points. Assume $W$ is normal. Then $f$ is an isomorphism.

**Corollary** Let $V \xrightarrow{f} W$ be a separable morphism that induces a bijection $V(k) \rightarrow W(k)$ on $k$-points. Then there exist stratifications of $V$ and $W$ into locally closed subvarieties

$$ V = \bigsqcup_{i=1}^{n} V_i \quad \text{resp.} \quad W = \bigsqcup_{i=1}^{n} W_i $$

such that $f$ restricts to an isomorphism $V_i \rightarrow W_i$ for $i = 1, 2, \ldots, n$. Hence $[V] = [W]$ in $SK(Var_k)$. 

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Corollary Let $V \xrightarrow{f} W$ be a separable morphism that induces a bijection $V(k) \to W(k)$ on $k$-points. Then $[V] = [W]$ in $SK(\text{Var}_k)$.

Corollary When $\text{char}(k)=0$,

$$SK(\text{Var}_k) \xrightarrow{\alpha_S} SK(\text{Constr}_k)$$

is into.
Remark If $\operatorname{char}(k)=0$ and $V \xrightarrow{f} W$ is a morphism of varieties that induces a bijection on $k$-points and is smooth at some point $x \in V$, then on an open neighborhood $U$ of $x$, $f|_U$ is an isomorphism.

When $\operatorname{char}(k)=0$, one can then use generic smoothness too to prove that $SK(Var_k) \xrightarrow{\alpha_S} SK(Constr_k)$ is injective.
Proposition

\[ SK(Var_k) \xrightarrow{\alpha_S} SK(Constr_k) \]

is not injective.

Enough to give a morphism of varieties \( f : X \to Y \) such that \([X] \neq [Y]\) in \( SK(Var_k) \) but \( f \) induces a bijection \( X(k) \to Y(k) \) on \( k \)-points, since this will ensure \( \alpha_S[X] = \alpha_S[Y] \) in \( SK(Constr_k) \).
Consider the diagram of schemes over $\mathbb{F}_p$

$$
\begin{array}{c}
X \\
\downarrow \downarrow \downarrow \\
S
\end{array}
\quad
\begin{array}{c}
\downarrow \\
\downarrow \\
S
\end{array}
\quad
\begin{array}{c}
\downarrow \downarrow \downarrow \\
X(p) \\
\downarrow \downarrow \downarrow \\
X
\end{array}

\quad
\begin{array}{c}
\downarrow \downarrow \downarrow \\
X(p) \\
\downarrow \downarrow \downarrow \\
X
\end{array}

\quad
\begin{array}{c}
\downarrow \downarrow \downarrow \\
S \\
\downarrow \downarrow \downarrow \\
S
\end{array}

where $Fr_p$ is the absolute Frobenius, $X^{(p)}$ is the pullback, and the relative Frobenius $Fr_{X/S}$ is the induced map into the pullback. When $S = \text{spec}(k)$ for a perfect field $k$ and $X$ is a variety over $k$, $Fr_{X/S}$ induces a bijection $X(k) \to X^{(p)}(k)$. 
Let \( k \) be algebraically closed of positive characteristic, and let \( E \) be an elliptic curve with \( j \)-invariant \( j_E \in k \). The Frobenius twist \( E^{(p)} \) of \( E \) has \( j \)-invariant \( (j_E)^p \). The relative Frobenius

\[
Fr : E \to E^{(p)}
\]

induces an isomorphism on \( k \)-points, so \( \alpha_S[E] = \alpha_S[E^{(p)}] \) in \( SK(Constr_k) \). If \( j_E \neq (j_E)^p \), then \( E \) and \( E^{(p)} \) are not isomorphic over \( k \). It follows that \([E] \neq [E^{(p)}]\) in \( SK(Var_k) \): two complete, irreducible curves represent the same class in \( SK(Var_k) \) iff they are isomorphic.
Theorem (Karzhemanov 2014; Borisov 2015)
Over the complex numbers, $\text{gr}(\text{var})$ is not injective.

In positive characteristics . . .

- is $\text{gr}(\text{var})$ injective? (probably not!)
- is $\alpha$ injective? (probably not!)
- is there a decent description of the kernel of $\alpha_S$?
  For example: it is the semiring congruence generated by those pairs $\langle X, Y \rangle$ where there exists a universal homeomorphism $X \to Y$.  

$SK(\text{Var}_k) \xrightarrow{\alpha_S} SK(\text{Constr}_k)$

$\text{gr}(\text{var}) \downarrow$

$K(\text{Var}_k) \xrightarrow{\alpha} K(\text{Constr}_k)$

$\text{gr}(\text{constr}) \downarrow$