The Grothendieck (semi)ring of algebraically closed fields

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model theory $\rightsquigarrow$ algebraic geometry

- First order logic: language contains relation, constant and function symbols; only allows $\forall$ and $\exists$ quantifiers ranging over the domain of interpretation
- 'Theory' is a set of first order axioms in a given language
- Can axiomatize most algebraic structures (e.g: ring, field, difference field; algebraically closed field, real closed field; category, groupoid; metric space) this way
- *No* obvious first order axiomatization for: noetherian ring; topological space; manifold; variety; ringed space; scheme; complete metric space

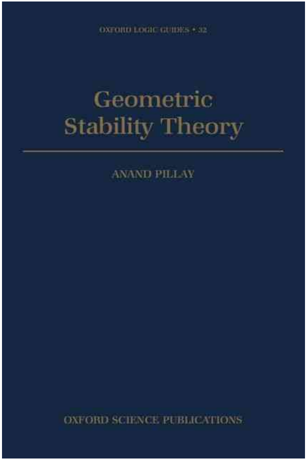
model theory, ca. 1950 – 1965

- Strong set-theoretical flavor. Motivating question: given a theory T and infinite cardinal κ, what is the cardinality of the set of isomorphism classes of models of T of size κ?
- Take T to be the theory of algebraically closed fields of a specific characteristic, κ an uncountable cardinal. Any two models of T of size κ are isomorphic (since they must have the same transcendence degree, namely κ, over the prime field). This is a rare phenomenon!
- Łós's conjecture: Let T contain countably many axioms. Suppose that for *at least one* uncountable κ, T has a unique isomorphism class of models of cardinality κ . Then, for *every* uncountable κ, T has a unique isomorphism class of models of cardinality κ.

model theory, ca. 1965 – 1990

- Morley (1966) proves Łós's conjecture. Major contribution: identifies a class of theories T whose models admit a structure theory of 'transcendence degree' similar to those of algebraically closed fields. (These theories are called ω-*stable*.)
- Vast technical elaboration of machinery of stability by Shelah, Baldwin, Lachlan, and others. Solution of the 'spectrum problem': *how many isomorphisms classes of models can a theory have in uncountable cardinalities*.
- Interest shifts to understanding the fine structure of specific models of specific types of theories.

model theory, ca. 1990 – 2000



geometry

A *model-theoretic geometry* consists of a set X ("points") and for each $n \in \mathbb{N}$, a set of subsets of X^n, denoted $\mathcal{B}_n$ (" definable subsets of X^n ") such that

- $\mathcal{B}_n$ is closed under boolean operations in X^n
- if $U \in \mathcal{B}_n$ and $V \in \mathcal{B}_m$ then $U \times V \in \mathcal{B}_{n+m}$
- if $U \in \mathcal{B}_n$ then $pr(U) \in \mathcal{B}_m$ for any projection $pr : X^n \to X^m$
- diagonals belong to $\mathcal{B}_n$; singletons belong to $\mathcal{B}_1$.

See van den Dries: *Tame geometry and o-minimal structures* for a minimal set of axioms.

associated category

Let X, $\mathcal{B}_n$, $n \in \mathbb{N}$ be a model-theoretic geometry. Introduce the category

$$\text{Def(X) with } \begin{cases} \text{objects} & = \text{ definable sets (i.e. elements of } \mathcal{B}_n) \\ \text{morphisms} & = \text{ definable functions} \end{cases}$$

i.e. a morphism from a definable $U \subseteq X^n$ to a definable $V \subseteq X^m$ is a function $f : U \to V$ whose graph belongs to $\mathcal{B}_{n+m}$.

geometry defined by a first-order language

Let $\mathcal{L}$ be a first-order signature (set of constant, function and relation symbols), and let the set X be equipped with interpretations of these symbols. For $U \subseteq X^n$, let

$$U \in \mathcal{B}_n \quad \textit{iff} \quad U = \{\, \mathbf{x} \in X^n \mid X \models \phi(\mathbf{x}) \,\}$$

for some first-order formula ϕ in the signature $\mathcal{L}$ (allowing parameters form X), with free variables from among the $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$.

motivating example: $\mathit{SemiLin}_{\mathbb{R}}$

underlying set: $\mathbb{R}$; *language:* $+$, scalar multiplication, $<$ $=$

- can define half-spaces $\{\mathbf{x} \mid \langle \mathbf{a}, \mathbf{x} \rangle < b\}$, affine subspaces and their finite boolean combinations; and via first order formulas, only these
- objects of $\mathit{SemiLin}_{\mathbb{R}}$ are finite boolean combinations of polytopes (possibly unbounded)
- morphisms of $\mathit{SemiLin}_{\mathbb{R}}$ are "piecewise linear" functions (i.e. set-functions whose graph belongs to $\mathit{SemiLin}_{\mathbb{R}}$; need not be continuous!)
- the n-simplex Δ_n and $[0, 1]^n$ are isomorphic in $\mathit{SemiLin}_{\mathbb{R}}$ (fun!)

motivating example: $FO(k)$

Let k be a field and let $FO(k)$ be the geometry of first order definable sets over k, in the language of $+ \quad \cdot \quad =$

Best understood when k is a local field, or an algebraically closed field (or "close" to being algebraically closed: pseudo-finite, pseudo-algebraically closed etc ...)

examples of geometries: $SemiAlg_{\mathbb{R}}$

Consider $FO(\mathbb{R})$. Relation $x < y$ is definable as $x \neq y \wedge \exists z(x + z^2 = y)$. So $FO(\mathbb{R})$ contains all semi-algebraic sets and in fact, coincides with semi-algebraic sets.

- subset of $\mathbb{R}^n$ is *semi-algebraic* if it can be written as a finite boolean combination of sets of the form $\{\mathbf{x} \mid p(\mathbf{x}) > 0\}$, where $p(\mathbf{x})$ is a polynomial
- projection of semi-algebraic set is semi-algebraic (Seidenberg), equivalently: the subset of $\mathbb{R}^n$ defined by any first order formula in the above language is semi-algebraic (Tarski)
- objects of $FO(\mathbb{R}) = SemiAlg_{\mathbb{R}}$ are semi-algebraic subsets of $\mathbb{R}^n$
- morphisms are set-functions with semi-algebraic graph (need not be continuous!)

examples of geometries: $Constr_k$

Let k be an algebraically closed field. $FO(k)$ will coincide with $Constr_k$, the category whose

- objects are constructible subsets of k^n (closed under projection by Chevalley's theorem; Tarski also proves that any subset of k^n definable via a first order formula in the above language, is constructible)
- morphisms are set-functions with constructible graph (need not be continuous!)

main problems concerning the category of definable sets

- Does it have quotients of equivalence relations?
- Find a notion of 'dimension' for objects
- Find a notion of 'size' for objects
- How do these invariants vary in families?
- Is there a *field* (perhaps even algebraically closed) among the objects of Def(X)?
- What are (abelian) group objects in Def(X)?

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- Find a notion of 'size' for objects
  (finitely additive) measure will be a homomorphism from the Grothendieck group to some abelian group; Euler characteristic will be homomorphism from the Grothendieck ring to some ring
- How do these invariants vary in families?
- Is there a *field* (perhaps even algebraically closed) among the objects of Def(X)?
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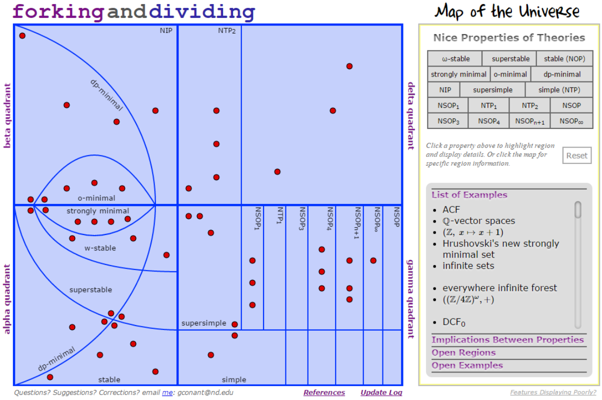
model theory, 2000 – present

Identify combinatorial conditions on first-order theories T that ensure 'nice solutions' to the main problems.

http://www.forkinganddividing.com



Proposition Def(X)

- has terminal object and pullbacks (so finite limits); they are computed as in Set
- has finite coproducts
- is *distributive*: the canonical maps

$$\varnothing \;\to\; X \times \varnothing$$

$$X \times Y \sqcup X \times Z \;\to\; X \times (Y \sqcup Z)$$

  are isomorphisms
- is boolean (subobject posets are boolean algebras; every subobject is a coproduct summand)

Grothendieck (semi)ring of a (small) distributive category $\mathcal{C}$

$SK(\mathcal{C})$ is the semiring whose elements are isomorphism classes $[X]$ of objects X, with $[X] \cdot [Y] := [X \times Y]$ and $[X] + [Y] := [X \sqcup Y]$.

$K(\mathcal{C})$ is the abelian group generated by isomorphism classes $[X]$ of objects X, with the relations $[X \sqcup Y] = [X] + [Y]$. Multiplication is induced by $[X] \cdot [Y] = [X \times Y]$.

- *Semiring* is a 'ring without additive inverses'.

- There are adjoint functors

$$\mathit{Ring} \underset{\mathit{inc}}{\overset{\mathit{groth}}{\leftrightarrows}} \mathit{SemiRing}$$

and $K(\mathcal{C}) = \mathit{groth}\big(SK(\mathcal{C})\big)$.

- Schanuel (1990) calls $SK(\mathcal{C})$ the "Burnside rig of $\mathcal{C}$" in his pioneering article *Negative sets have Euler characteristic and dimension*.

Theorem (Lojasiewicz; Hironaka, ca. 1960) The inclusion of categories $\mathit{SemiLin}_{\mathbb{R}} \hookrightarrow \mathit{SemiAlg}_{\mathbb{R}}$ induces isomorphisms

$$SK(\mathit{SemiLin}_{\mathbb{R}}) \overset{=}{\to} SK(\mathit{SemiAlg}_{\mathbb{R}})$$

Theorem (Schanuel 1990)
$SK(\mathit{SemiLin})$ is a finitely presentable semiring, isomorphic to

$$\mathbb{N}[X]/(X = 2X + 1).$$

Theorem (Schanuel 1990; o-minimal: van den Dries, 1998)
There is a monomorphism

$$SK(\mathit{SemiAlg}) \xrightarrow{\text{dim} \times \text{eu}} \{\mathbb{N} \cup -\infty\}_{\langle +, \text{max} \rangle} \times \mathbb{Z}$$

where *dim* is topological dimension and *eu* is the combinatorial Euler characteristic (to be defined momentarily).

Schanuel's presentation

Commutative diagram

$$\mathbb{N}[X]/(X = 2X+1) \xrightarrow{deg \times \left\{(f(X) \mapsto f(-1)\right\}} \mathbb{N}_{-\infty} \times \mathbb{Z}$$

$$\mathbb{N}[X]/(X = 2X+1) \xrightarrow{X \mapsto (0,1)} SK(SemiAlg) \xrightarrow{dim \times eu} \mathbb{N}_{-\infty} \times \mathbb{Z}$$

Degree-wise induction shows top arrow injective; left arrow surjective, hence isomorphism.

from semiring to ring

Commutative diagram

$$\begin{array}{ccc} SK(SemiAlg) & \xrightarrow{\dim \times \mathrm{eu}} & \{\mathbb{N} \cup -\infty\} \times \mathbb{Z} \\ \downarrow{\scriptstyle groth} & & \downarrow{\scriptstyle groth} \\ K(SemiAlg) & \xrightarrow{\mathrm{eu}} & \mathbb{Z} \end{array}$$

groth preserves products. It need not preserve monos, but an easy argument shows that in the present case, the bottom arrow is an isomorphism.

the combinatorial Euler characteristic

Let X be semi-algebraic and $(V, \mathcal{S})$ an open-cell complex such that X is semi-algebraically homeomorphic to $|\mathcal{S}|$.

Definition $\mathrm{eu}(X) = \sum_{U \in \mathcal{S}} (-1)^{\dim(U)}$

Theorem $\mathrm{eu}(X)$ is independent of the open-cell decomposition chosen.

The proof needs that any two semi-algebraic open-cell decompositions have a common semi-algebraic refinement.

Euler-Poincaré characteristic

Let $\mathbb{F}$ be any field. Let $H^*(-;\mathbb{F})$ denote sheaf (or equivalently, singular) cohomology and let $H_c^*(-;\mathbb{F})$ denote cohomology with compact support. Let X be a semi-algebraic set.

If X is locally compact,

$$\mathrm{eu}(X) = \chi_c(X) = \sum_{i=0}^{\dim(X)} (-1)^i \dim_{\mathbb{F}} H_c^i(X;\mathbb{F}).$$

Follows from H_c^* long exact sequence of $U \subset X$ where U is open, X Hausdorff, locally compact; cell decomposition of X and induction.

embarrassing!

- Is $\mathrm{eu}(X) = \chi_c(X)$ for *all* semi-algebraic X, not just locally compact ones? Is there a cohomological interpretation of $\mathrm{eu}(X)$ valid for all X?
- χ_c is a proper (topological) homotopy invariant. Is that true for $\mathrm{eu}(X)$ as well?

Theorem (TB, 2011) If X, Y are semi-algebraic (or more generally, o-minimal, belonging to possibly two distinct o-minimal structures) and topologically homeomorphic then $eu(X) = eu(Y)$.

Proof reduces to locally compact case with the help of an intrinsically defined stratification of o-minimal sets.

Remark Already two polyhedra can be topologically homeomorphic but not semi-algebraically so (Milnor, counterexample to the polyhedral Hauptvermutung, 1961).

non-archimedean example

Recall that $FO(\mathbb{Q}_p)$ is the geometry of subsets of $(\mathbb{Q}_p)^n$ that are first order definable in the language $+ \cdot =$.

Theorem (Clucker–Haskell 2000) The Grothendieck ring of $FO(\mathbb{Q}_p)$ is trivial.

When $p \neq 2$

$$\mathbb{Z}_p = \{ \; x \in \mathbb{Q}_p \mid \exists y \in \mathbb{Q}_p (y^2 = 1 + px^2) \; \}$$

When $p = 2$

$$\mathbb{Z}_2 = \{ \; x \in \mathbb{Q}_2 \mid \exists y \in \mathbb{Q}_2 (y^3 = 1 + 2x^3) \; \}$$

So $\mathbb{Z}_p$ is an object of $FO(\mathbb{Q}_p)$.

non-archimedean example

$\mathbb{Z}_p$ is an object of $FO(\mathbb{Q}_p)$. Clucker and Haskell then give an explicit bijection between $\mathbb{Z}_p - \{0\}$ and $\mathbb{Z}_p$ in $FO(\mathbb{Q}_p)$.

In any distributive category $\mathcal{C}$, if for some object Z the objects $Z - \{pt\}$ and Z are isomorphic, then $[pt] = [\varnothing]$ in $K(\mathcal{C})$, so the Grothendieck ring $K(\mathcal{C})$ is trivial.

Grothendieck's Grothendieck ring of varieties

For any field k, let $SK(Var_k)$ be the semiring with generators the varieties over k and relations

$$[X] = [Y] \quad \text{if } X \text{ and } Y \text{ are isomorphic over } k$$

$$[X] = [X - U] + [U]$$

for every open subvariety U of X with complement $X - U$.

The product of $[X]$ and $[Y]$ is $[X \otimes_k Y]$.

$K(Var_k)$ is the ring generated by the same generators and relations.

back to $FO(k) = Constr_k$

Let k be algebraically closed. There's a natural homomorphism

$$SK(Var_k) \xrightarrow{\alpha_S} SK(Constr_k)$$

Theorem (TB 2013) α_S is an isomorphism when $char(k) = 0$. It is surjective but not injective when $char(k) > 0$.

Corollary (folk; Sebag-Nicaise 2011) The model-theorist's Grothendieck ring of the field k and Grothendieck's Grothendieck ring of varieties over k, are isomorphic for k algebraically closed of characteristic 0.

the comparison homomorphism

$$SK(Var_k) \xrightarrow{\alpha_S} SK(Constr_k)$$

Given variety X, choose a decomposition (as point set) $X = \bigsqcup_{i=1}^{n} C_i$ into pairwise disjoint affine constructible sets and send $[X]$ to $\sum_{i=1}^{n}[C_i] \in SK(Constr_k)$.

- such decompositions always exist; e.g. choose an affine atlas $\{U_i \mid i = 1, 2, 3, \dots, n\}$ and set

$$C_i := U_i - (\sum_{j=1}^{i-1} U_j)$$

- $\sum_{i=1}^{n}[C_i]$, as element of $SK(Constr_k)$, is independent of decomposition chosen
- α_S is compatible with $+$ and $\times$.

surjectivity

Proposition

$$SK(Var_k) \xrightarrow{\alpha_S} SK(Constr_k)$$

is onto (in every characteristic).

Use the fact that any constructible subset of $\mathbb{A}^n_k$ can be stratified as a disjoint union of locally closed subvarieties of $\mathbb{A}^n_k$.

role of separability

Key fact (cf. Zariski's main theorem) Let V, W be irreducible varieties and $V \xrightarrow{f} W$ a separable morphism that induces a bijection on k-points. Assume W is normal. Then f is an isomorphism.

Corollary Let $V \xrightarrow{f} W$ be a separable morphism that induces a bijection $V(k) \to W(k)$ on k-points. Then there exist stratifications of V and W into locally closed subvarieties

$$V = \bigsqcup_{i=1}^{n} V_i \quad \text{resp.} \quad W = \bigsqcup_{i=1}^{n} W_i$$

such that f restricts to an isomorphism $V_i \to W_i$ for $i = 1, 2, \ldots, n$. Hence $[V] = [W]$ in $SK(Var_k)$.

injectivity

Corollary Let $V \xrightarrow{f} W$ be a separable morphism that induces a bijection $V(k) \to W(k)$ on k-points. Then $[V] = [W]$ in $SK(Var_k)$.

Corollary When char(k)=0,

$$SK(Var_k) \xrightarrow{\alpha_S} SK(Constr_k)$$

is into.

role of smoothness

Remark If char(k)=0 and $V \xrightarrow{f} W$ is a morphism of varieties that induces a bijection on k-points and is smooth at some point $x \in V$, then on an open neighborhood U of x, $f|_U$ is an isomorphism.

When char(k)=0, one can then use generic smoothness too to prove that $SK(Var_k) \xrightarrow{\alpha_S} SK(Constr_k)$ is injective.

positive characteristics

Proposition

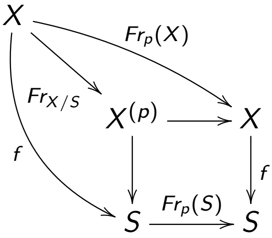
$$SK(Var_k) \xrightarrow{\alpha_S} SK(Constr_k)$$

is not injective.

Enough to give a morphism of varieties $f : X \to Y$ such that $[X] \neq [Y]$ in $SK(Var_k)$ but f induces a bijection $X(k) \to Y(k)$ on k-points, since this will ensure $\alpha_S[X] = \alpha_S[Y]$ in $SK(Constr_k)$.

relative Frobenius

Consider the diagram of schemes over $\mathbb{F}_p$



where Fr_p is the absolute Frobenius, $X^{(p)}$ is the pullback, and the *relative Frobenius* $Fr_{X/S}$ is the induced map into the pullback. When $S = spec(k)$ for a perfect field k and X is a variety over k, $Fr_{X/S}$ induces a bijection $X(k) \to X^{(p)}(k)$.

twists of elliptic curves

Let k be algebraically closed of positive characteristic, and let E be an elliptic curve with j-invariant $j_E \in k$. The Frobenius twist $E^{(p)}$ of E has j-invariant $(j_E)^p$. The relative Frobenius

$$Fr : E \to E^{(p)}$$

induces an isomorphism on k-points, so $\alpha_S[E] = \alpha_S[E^{(p)}]$ in $SK(Constr_k)$. If $j_E \neq (j_E)^p$, then E and $E^{(p)}$ are not isomorphic over k. It follows that $[E] \neq [E^{(p)}]$ in $SK(Var_k)$: two complete, irreducible curves represent the same class in $SK(Var_k)$ iff they are isomorphic.

known and unknown

$$\begin{array}{ccc} SK(Var_k) & \xrightarrow{\alpha_S} & SK(Constr_k) \\ \downarrow{\scriptstyle gr(var)} & & \downarrow{\scriptstyle gr(constr)} \\ K(Var_k) & \xrightarrow{\alpha} & K(Constr_k) \end{array}$$

Theorem (Karzhemanov 2014; Borisov 2015)
Over the complex numbers, gr(var) is not injective.

In positive characteristics ...

- is gr(var) injective? (probably not!)
- is α injective? (probably not!)
- is there a decent description of the kernel of α_S?
  For example: it is the semiring congruence generated by those pairs $\langle X, Y \rangle$ where there exists a universal homeomorphism $X \to Y$.