Theorem (L. E. J. Brouwer, 1910)

Let $B_n$ be the $n$-dimensional unit ball $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Let $f : B_n \to B_n$ be any continuous function. Then $f(x) = x$ for some point $x \in B_n$. 

Is this obvious or what? No. Is it even believable? I wouldn't say so!
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Brouwer’s sensory intuitions

The theorem is supposed to have originated from Brouwer’s observation of a cup of coffee. If one stirs to dissolve a lump of sugar, it appears there is always a point without motion. He drew the conclusion that at any moment, there is a point on the surface that is not moving.

Brouwer is said to have added: “I can formulate this splendid result different, I take a horizontal sheet, and another identical one which I crumple, flatten and place on the other. Then a point of the crumpled sheet is in the same place as on the other sheet.”
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Me: I’ve never been less convinced in my life.
what we’ll do

▶ Find a variant of the theorem that is visually compelling.
what we’ll do

- Find a variant of the theorem that is visually compelling.
- Try to turn that into a proof. Still hard!
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Tibor Beke
what we’ll do

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- Simplify the geometry *drastically*. Turn it into pure combinatorics. Guess the (?) statement of the *sign pattern theorem*.
- Prove the sign pattern theorem. Surprise (?) the only proof I know is algebraic topological at heart.
what we’ll do

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- Simplify the geometry drastically. Turn it into pure combinatorics. Guess the (?) statement of the sign pattern theorem.
- Prove the sign pattern theorem. Surprise (?): the only proof I know is algebraic topological at heart.
- Use approximate fixed point theory and compactness to conclude Brouwer’s fixed point theorem in topology.
Let $f : [0, 1] \to [0, 1]$ be continuous. Consider $g(x) = f(x) - x$. 
$g(0) \geq 0$ and $g(1) \leq 0$.

By the Intermediate Value Theorem, $g(x_0) = 0$ for some $x_0 \in [0, 1]$. Thence $f(x_0) = x_0$. 

one-dimensional case is easy
Unit disk $B_2$ and unit square $[0, 1]^2$ are **homeomorphic** (there’s a bijection between them that is continuous, with a continuous inverse).
Unit disk $B_2$ and unit square $[0, 1]^2$ are *homeomorphic* (there’s a bijection between them that is continuous, with a continuous inverse).

**Challenge** Say that a topological space $X$ has the *fixed point property* if any continuous map $f : X \to X$ has a fixed point.

Suppose $X$ and $Y$ are homeomorphic. Then if $X$ has the fixed point property, so does $Y$. 
We want to prove: any continuous map
\((f_1(x, y), f_2(x, y)) : [0, 1]^2 \rightarrow [0, 1]^2\) has a fixed point.
two dimensions, take 2

We want to prove: any continuous map
\( \langle f_1(x, y), f_2(x, y) \rangle : [0, 1]^2 \to [0, 1]^2 \) has a fixed point.

Exhibiting a *total* lack of imagination, we’ll do the same as in the one-dimensional case:

Let \( g_1(x, y) = f_1(x, y) - x \) and \( g_2(x, y) = f_2(x, y) - y \).

Then \( g_1, g_2 \) are continuous functions \([0, 1]^2 \to \mathbb{R}\).

For all \( 0 \leq y \leq 1 \), \( g_1(0, y) \geq 0 \) and \( g_1(1, y) \leq 0 \).

For all \( 0 \leq x \leq 1 \), \( g_2(x, 0) \geq 0 \) and \( g_2(x, 1) \leq 0 \).
two dimensions, take 2

We want to prove: any continuous map
\[ \langle f_1(x, y), f_2(x, y) \rangle : [0, 1]^2 \to [0, 1]^2 \] has a fixed point.

Exhibiting a *total* lack of imagination, we’ll do the same as in the one-dimensional case:

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For all \( 0 \leq y \leq 1 \), \( g_1(0, y) \geq 0 \) and \( g_1(1, y) \leq 0 \).

For all \( 0 \leq x \leq 1 \), \( g_2(x, 0) \geq 0 \) and \( g_2(x, 1) \leq 0 \).

A point \( \langle x, y \rangle \) that is a simultaneous zero of \( g_1 \) and \( g_2 \) is the same as a fixed point of \( \langle f_1(x, y), f_2(x, y) \rangle \).
Suppose $g : [0, 1]^2 \to \mathbb{R}$ is continuous such that $g(x, 0) \geq 0$ and $g(x, 1) \leq 0$ for all $0 \leq x \leq 1$. 
Suppose $g : [0, 1]^2 \rightarrow \mathbb{R}$ is continuous such that $g(x, 0) \geq 0$ and $g(x, 1) \leq 0$ for all $0 \leq x \leq 1$.

Such a function "ought to" have a "band of zeros" connecting the $x = 0$ and $x = 1$ edges:
Suppose $g : [0, 1]^2 \to \mathbb{R}$ is continuous such that $g(x, 0) \geq 0$ and $g(x, 1) \leq 0$ for all $0 \leq x \leq 1$.

If the zero locus of $g$ did not have a component connecting the $x = 0$ and $x = 1$ edges.
If the **zero locus** of $g$ did not have a component connecting the $x = 0$ and $x = 1$ edges then there would exist a zero-free path connecting a negative value of $g$ with a positive value, contradicting the 1-dimensional case of the Intermediate Value Theorem!
why Brouwer’s theorem is plausible

Suppose $g_1, g_2$ are continuous functions $[0, 1]^2 \to \mathbb{R}$ such that for all $0 \leq y \leq 1$, $g_1(0, y) \geq 0$ and $g_1(1, y) \leq 0$ for all $0 \leq x \leq 1$, $g_2(x, 0) \geq 0$ and $g_2(x, 1) \leq 0$.

Then:

- there is a path in the zero locus of $g_1$ connecting the $y = 0$ and $y = 1$ edges
- there is a path in the zero locus of $g_2$ connecting the $x = 0$ and $x = 1$ edges
why Brouwer’s theorem is plausible

path in the zero locus of \( g_1 \) and path in the zero locus of \( g_2 \)

- if you connect top to bottom and left to right, those paths “must” intersect
- a common zero of \( g_1 \) and \( g_2 \) exists
- Brouwer fixed point theorem for \([0, 1]^2\) follows!
just intuitively

Suppose \( g : [0, 1]^2 \rightarrow \mathbb{R} \) is continuous such that \( g(x, 0) \geq 0 \) and \( g(x, 1) \leq 0 \) for all \( 0 \leq x \leq 1 \).

Such a function “ought to” have a “band of zeros” connecting the \( x = 0 \) and \( x = 1 \) edges:
Suppose $g : [0, 1]^2 \to \mathbb{R}$ is continuous such that $g(x, 0) \geq 0$ and $g(x, 1) \leq 0$ for all $0 \leq x \leq 1$.

Let $Z = g^{-1}(0) \subseteq [0, 1]^2$.

Then there exists a continuous path $\langle p_1, p_2 \rangle : [0, 1] \to Z$ such that $p_1(0) = 0$ and $p_1(1) = 1$. 

:-(
Suppose $g : [0, 1]^2 \rightarrow \mathbb{R}$ is continuous such that $g(x, 0) \geq 0$ and $g(x, 1) \leq 0$ for all $0 \leq x \leq 1$.

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This is false.
Suppose $g : [0, 1]^2 \to \mathbb{R}$ is continuous such that $g(x, 0) \geq 0$ and $g(x, 1) \leq 0$ for all $0 \leq x \leq 1$.

Let $Z = g^{-1}(0) \subseteq [0, 1]^2$.

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This is false.

**Challenge** Give a counterexample.
Suppose \( g : [0, 1]^2 \to \mathbb{R} \) is continuous such that 
\[ g(x, 0) \geq 0 \quad \text{and} \quad g(x, 1) \leq 0 \quad \text{for all} \quad 0 \leq x \leq 1. \]
Let \( Z = g^{-1}(0) \subseteq [0, 1]^2. \)
Suppose $g : [0, 1]^2 \to \mathbb{R}$ is continuous such that $g(x, 0) \geq 0$ and $g(x, 1) \leq 0$ for all $0 \leq x \leq 1$. Let $Z = g^{-1}(0) \subseteq [0, 1]^2$.

Then there is a connected component $Z_0$ of $Z$ that intersects both $x = 0$ and $x = 1$. 
Suppose $g : [0, 1]^2 \to \mathbb{R}$ is continuous such that $g(x, 0) \geq 0$ and $g(x, 1) \leq 0$ for all $0 \leq x \leq 1$.

Let $Z = g^{-1}(0) \subseteq [0, 1]^2$.

- Then there is a connected component $Z_0$ of $Z$ that intersects both $x = 0$ and $x = 1$. **True.**
Suppose $g : [0, 1]^2 \to \mathbb{R}$ is continuous such that $g(x, 0) \geq 0$ and $g(x, 1) \leq 0$ for all $0 \leq x \leq 1$.

Let $Z = g^{-1}(0) \subseteq [0, 1]^2$.

- Then there is a connected component $Z_0$ of $Z$ that intersects both $x = 0$ and $x = 1$. True.
- Then there is a path connected component $Z_0$ of $Z$ that intersects both $x = 0$ and $x = 1$. False.
Suppose \( g : [0, 1]^2 \rightarrow \mathbb{R} \) is continuous such that \( g(x, 0) \geq 0 \) and \( g(x, 1) \leq 0 \) for all \( 0 \leq x \leq 1 \).

Let \( Z = g^{-1}(0) \subseteq [0, 1]^2 \).

- Then there is a connected component \( Z_0 \) of \( Z \) that intersects both \( x = 0 \) and \( x = 1 \). True.
- Then there is a path connected component \( Z_0 \) of \( Z \) that intersects both \( x = 0 \) and \( x = 1 \). False.
Hairy!
cannot always find cross-section of the zero locus that “looks like” an interval

- topology gets yet more complicated for $[0, 1]^n$ with $n > 2$
- proof can be completed this way but needs difficult algebraic topological machinery — more difficult than other (algebraic topological) proofs of Brouwer’s fixed point theorem
Suppose $\mathcal{F}$ is a set of continuous functions from $[0, 1]^n$ to itself that is *uniformly dense* among all continuous functions from $[0, 1]^n$ to itself:

For any continuous $g : [0, 1]^n \rightarrow [0, 1]^n$ and $\epsilon > 0$ there exists $f \in \mathcal{F}$ such that $\|g(x) - f(x)\| < \epsilon$ for all $x \in [0, 1]^n$.

Suppose you manage to prove: any $f \in \mathcal{F}$ has a fixed point.

Then it follows that any continuous $[0, 1]^n \rightarrow [0, 1]^n$ has a fixed point.
moral

- find a ‘nice’ class of continuous functions
- should be uniformly dense among all continuous functions but not allow hairy point-set theoretic phenomena
- prove the intermediate value theorem on $[0, 1]^n$ for these functions
- Brouwer’s fixed point theorem follows for all continuous functions by approximate fixed point theory.
What are ‘nice’ families of functions, dense among all continuous real-valued functions on $[0, 1]^n$?
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- polynomials in $n$ variables
What are ‘nice’ families of functions, dense among all continuous real-valued functions on $[0, 1]^n$?

- polynomials in $n$ variables
- trigonometric polynomials
nice?

What are ‘nice’ families of functions, dense among all continuous real-valued functions on $[0, 1]^n$?

- polynomials in $n$ variables
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- piecewise linear functions
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- step functions.
### Brouwer Fixed Point Theorem

The sign pattern theorem

#### Algebraic Topology

**Statement of the Theorem**  
Making it plausible  
Taming the topology

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Tibor Beke  
**The Sign Pattern Theorem**
### Brouwer Fixed Point Theorem

#### Statement of the Theorem

Making it plausible through taming the topology.

#### Algebraic Topology

The sign pattern theorem provides a method to visualize and understand the fixed point theorem in algebraic terms.

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This grid illustrates the sign pattern associated with the Brouwer fixed point theorem, allowing for easier visualization of the theorem's conditions and implications.
sign patterns in matrices of signs

We’ll be interested in $n \times m$ matrices, each entry of which contains two symbols:

either $+$ or $-$ (corresponding to the sign of $g_1$) as well as
either $+$ or $-$ (corresponding to the sign of $g_2$)

such that

boundary conditions \[
\begin{cases}
\text{the first column must contain } + \\
\text{the last column must contain } - \\
\text{the bottom row must contain } + \\
\text{the top row must contain } - .
\end{cases}
\]
<table>
<thead>
<tr>
<th>Tibor Beke</th>
<th>the sign pattern theorem</th>
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<tbody>
<tr>
<td>brouwer fixed point theorem</td>
<td>seeking a formulation</td>
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<tr>
<td>the sign pattern theorem</td>
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Tibor Beke
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<tr>
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What are the “simultaneous zeros” of the two functions, signified by the red resp. blue sign entries?

Simultaneous zeros sorta kinda like correspond to “adjacent entries” of the sign matrix where “sign changes occur”.

What does this mean, really?
Consider a string of + and − signs beginning with + and ending with −

\[+ + − + + − + + − −\]

Then
Consider a string of $+$ and $-$ signs beginning with $+$ and ending with $-$

$+$ $+$ $-$ $+$ $+$ $-$ $+$ $+$ $-$

Then

- the string contains $+_-$ as substring somewhere
Consider a string of $+$ and $-$ signs beginning with $+$ and ending with $-$.

Then

- the string contains $+ -$ as substring somewhere
- the string contains an *odd* number of sign changes.
Consider a string of + and − signs beginning with + and ending with −.

Then

- the string contains $+−$ as substring somewhere
- the string contains an $odd$ number of sign changes
- the number of $+−$ substrings is one more than the number of $−+$ substrings.
Any sign matrix satisfying the boundary conditions, like:

```
+-  +-  --  +-  --  --
+-  --  +-  --  ++  --
+-  ++  --  ++  --  --
++  --  +-  --  --  +-  
++  --  --  ++  ++  --
++  --  --  ++  ++  --
```
Any sign matrix satisfying the boundary conditions
Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries

$\begin{bmatrix} ++ & +- \\ +- & -- \end{bmatrix}$ and $\begin{bmatrix} ++ & -+ \\ -+ & -- \end{bmatrix}$ (?)

(b) contains adjacent $++$ and $--$ entries (?)

(c) contains adjacent $++$ and $--$ entries (?)

(d) contains adjacent sign-reversed entries, that is, adjacent $++$ and $--$ or adjacent $++$ and $--$ (?)

(e) contains a $2 \times 2$ submatrix where all four symbols $+ - + -$ occur (?)
Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries $++$, $+-$, $-+$ and $--$ (?)

(b) contains adjacent $++$ and $--$ entries (?)
Any sign matrix satisfying the boundary conditions
(a) contains a $2 \times 2$ submatrix with all four types of entries $++$, $+-$, $-+$ and $--$ (?)
(b) contains adjacent $++$ and $--$ entries (?)
(c) contains adjacent $+-$ and $-+$ entries (?)
Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries $++$, $+-$, $-+$, and $--$.

(b) contains adjacent $++$ and $--$ entries.

(c) contains adjacent $+-$ and $-+$ entries.

(d) contains adjacent sign-reversed entries, that is, adjacent $++$ and $--$ or adjacent $+-$ and $-+$.
Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries

\[
\begin{array}{cccc}
++ & +- & -+ & -- \\
\end{array}
\] (?)

(b) contains adjacent $++$ and $--$ entries (?)

(c) contains adjacent $+-$ and $-+$ entries (?)

(d) contains adjacent sign-reversed entries, that is, adjacent $++$ and $--$ or adjacent $+-$ and $-+$ (?)

(e) contains a $2 \times 2$ submatrix where all four symbols $+-+-$ occur (?)
finding a statement to prove

Any sign matrix satisfying the boundary conditions
Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries

$\begin{pmatrix}++ & \pm \mp \\ \pm \mp & -\pm \mp \end{pmatrix}$ and $\begin{pmatrix}++ & \pm \pm \\ \pm \pm & -\pm \pm \end{pmatrix}$
Any sign matrix satisfying the boundary conditions

(a) contains a 2 \times 2 submatrix with all four types of entries

\[ \begin{array}{cc} ++ & + - \\ - + & - - \end{array} \]

nope
finding a statement to prove

Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries

\[
\begin{array}{ccc}
  + & - & - \\
  + & - & +
\end{array}
\]

nope

(b) contains adjacent \[++] and \[--\] entries

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finding a statement to prove

Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries $++$, $+-$, $-+$ and $--$ nope

(b) contains adjacent $+-$ and $-+$ entries nope
finding a statement to prove

Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries

(b) contains adjacent $++$ and $--$ entries

(c) contains adjacent $+-$ and $-+$ entries

(d) contains adjacent sign-reversed entries, that is, adjacent $++$ and $--$ or adjacent $+-$ and $-+$ entries, yes, in dimension 2 at least

(e) contains a $2 \times 2$ submatrix where all four symbols $+, -, +, -$ occur YES in all dimensions.
finding a statement to prove

Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries

++ , +- , -+ and -- nope

(b) contains adjacent ++ and -- entries nope

(c) contains adjacent +- and -+ entries nope
finding a statement to prove

Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries

$\begin{array}{cc}
++ & +-\\
+- & -+
\end{array}$

and $\begin{array}{cc}
-+ & --
\end{array}$

nope

(b) contains adjacent $\begin{array}{cc}
++ & --
\end{array}$ entries

nope

(c) contains adjacent $\begin{array}{cc}
+- & --
\end{array}$ entries

nope

(d) contains adjacent sign-reversed entries, that is, adjacent $\begin{array}{cc}
++ & --
\end{array}$ or adjacent $\begin{array}{cc}
+- & --
\end{array}$
finding a statement to prove

Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries $\begin{pmatrix} ++ \\ - - \end{pmatrix}$, $\begin{pmatrix} + - \\ - + \end{pmatrix}$ and $\begin{pmatrix} - - \\ + + \end{pmatrix}$ nope

(b) contains adjacent $\begin{pmatrix} ++ \\ - - \end{pmatrix}$ and $\begin{pmatrix} - - \\ + + \end{pmatrix}$ entries nope

(c) contains adjacent $\begin{pmatrix} + - \\ - + \end{pmatrix}$ entries nope

(d) contains adjacent sign-reversed entries, that is, adjacent $\begin{pmatrix} ++ \\ - - \end{pmatrix}$ and $\begin{pmatrix} - - \\ + + \end{pmatrix}$ or adjacent $\begin{pmatrix} + - \\ - + \end{pmatrix}$ and $\begin{pmatrix} - + \\ + - \end{pmatrix}$ yes, in dimension 2 at least

Tibor Beke
finding a statement to prove

Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries

\[
\begin{array}{ccc}
++ & , & -- \\
+− & , & −+ \\
−+ & , & −− \\
\end{array}
\]

nope

(b) contains adjacent $++$ and $−−$ entries

nope

(c) contains adjacent $+−$ and $−+$ entries

nope

(d) contains adjacent sign-reversed entries, that is, adjacent $++$ and $−−$ or adjacent $+−$ and $−+$

yes, in dimension 2 at least

(e) contains a $2 \times 2$ submatrix where all four symbols $+−+−$ occur
finding a statement to prove

Any sign matrix satisfying the boundary conditions

(a) contains a $2 \times 2$ submatrix with all four types of entries

| ++ | +− | −+ | −− |

| yes | no | no | no |

(b) contains adjacent $++$ and $−−$ entries

| yes | no |

(c) contains adjacent $+−$ and $−+$ entries

| yes | no |

(d) contains adjacent sign-reversed entries, that is, adjacent

| ++ | +− | −+ | −− |

| yes, in dimension 2 at least |

(e) contains a $2 \times 2$ submatrix where

| ++ | +− | −+ | −− |

| YES in all dimensions. |
Any sign matrix satisfying the boundary conditions will contain $2 \times 2$ submatrix where all four symbols $+ - - +$ occur, such as

\[
\begin{array}{cc}
+ & - \\
- & + \\
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
+ & + \\
- & - \\
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
+ & + \\
- & - \\
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
- & + \\
- & + \\
\end{array}.
\]
Any sign matrix satisfying the boundary conditions will contain $2 \times 2$ submatrix where all four symbols $+ - + -$ occur, such as

\[
\begin{array}{cc}
  + & - \\
  - & + \\
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
  + & + \\
  - & - \\
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
  + & + \\
  - & - \\
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
  - & - \\
  + & + \\
\end{array}.
\]

▶ can in fact guarantee that all four symbols will be found in two adjacent entries (possibly “corner adjacent”)
Any sign matrix satisfying the boundary conditions will contain $2 \times 2$ submatrix where all four symbols $+ - + -$ occur, such as

\[
\begin{array}{cc}
+ & - \\
- & +
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
+ & + \\
- & -
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
+ & + \\
- & -
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
- & + \\
- & +
\end{array}.
\]

- can in fact guarantee that all four symbols will be found in two adjacent entries (possibly “corner adjacent”)
- don’t know if this holds in dimensions greater than two!
Any sign matrix satisfying the boundary conditions will contain $2 \times 2$ submatrix where all four symbols $+ - + -$ occur, such as

\[
\begin{array}{cc}
+ & - \\
- & + \\
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
+ & + \\
- & - \\
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
+ & + \\
- & - \\
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
- & + \\
- & + \\
\end{array}.
\]

- can in fact guarantee that all four symbols will be found in two adjacent entries (possibly “corner adjacent”)
- don’t know if this holds in dimensions greater than two!
- can prove that in dimension $n$, all $2n$ symbols will be found in $n + 1$ adjacent $n$-cells.
Two approaches to the sign pattern theorem:

- digital: direct, visual, combinatorial, hard
- algebraic: indirect “magic”, non-constructive, powerful
digital topology
digital divide
digital crossing
adjacent entries containing reversed sign-pairs
Challenge Make the above argument precise.

Challenge++ Make the above argument work in 3 dimensions.
assign to each sign-pair one of three labels

\[
\begin{align*}
  a \{ & \quad ++ \\
  b \{ & \quad +− \\
  c \{ & \quad −+ \\
  \} & \quad −− 
\end{align*}
\]
setting up the algebra

given a sign matrix

\[
\begin{array}{cccccc}
+ & - & - & - & + & - \\
+ & + & - & - & - & + \\
+ & - & + & + & + & - \\
+ & + & + & - & + & - \\
+ & + & + & - & + & - \\
\end{array}
\]
replace each sign-pair by its label

b c c b c
a b c c c
b a b a c
a a c a c
setting up the algebra

note new boundary conditions
consider the dual grid, placing the symbols at the vertices and connecting them with edges

```
 b --- c --- c --- b --- c
  |     |     |     |
 a --- b --- c --- c --- c
  |     |     |     |
 b --- a --- b --- a --- c
  |     |     |     |
 a --- a --- c --- a --- c
```
setting up the algebra

triangulate the grid in \textit{any} way, obtaining a simplicial complex with labeled vertices

\[
\begin{array}{c}
\text{b} \quad \text{c} \quad \text{c} \quad \text{b} \quad \text{c} \\
\text{a} \quad \text{b} \quad \text{c} \quad \text{c} \quad \text{c} \\
\text{a} \quad \text{b} \quad \text{a} \quad \text{b} \quad \text{a} \quad \text{c} \\
\text{a} \quad \text{a} \quad \text{a} \quad \text{c} \quad \text{a} \quad \text{c}
\end{array}
\]
Let $E$ (for “edges”) be the abelian group generated by the symbols $\langle x, y \rangle$ where $x, y \in \{a, b, c\}$, subject to the relations

$$\langle x, y \rangle = -\langle y, x \rangle$$

$$\langle x, x \rangle = 0$$

for all $x, y \in \{a, b, c\}$.
Note that all triangles in the labeled complex can be oriented compatibly (say, clockwise). Let the boundary operator $\partial$ be the map from labeled, oriented triangles to $E$ defined by

$$\partial \left( \begin{array}{c} x \\ y \\ z \\ \end{array} \right) = \langle x, y \rangle + \langle y, z \rangle + \langle z, x \rangle$$
the labeled boundary operator

examples

\[ \partial \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \langle a, c \rangle + \langle c, b \rangle + \langle b, a \rangle \]

\[ = -\langle a, b \rangle - \langle b, c \rangle - \langle c, a \rangle \]

\[ \partial \begin{pmatrix} c \\ a \\ c \end{pmatrix} = \langle a, c \rangle + \langle c, c \rangle + \langle c, a \rangle \]

\[ = \langle a, c \rangle + 0 - \langle a, c \rangle = 0 \]
labeled boundaries: the key property

A triangle is **well-labeled** if all three labels $a, b, c$ show up on its vertices. It is **positive well-labeled** if $a, b, c$ occur clockwise and **negative well-labeled** if they occur in the other orientation.

**Lemma.** Let $T$ be a colored triangle.

- $\partial(T) = 0$ unless $T$ is well-labeled
- $\partial(T) = \langle a, b \rangle + \langle b, c \rangle + \langle c, a \rangle$ if $T$ is positive well-labeled
- $\partial(T) = -\langle a, b \rangle - \langle b, c \rangle - \langle c, a \rangle$ if $T$ is negative well-labeled.
proof of the sign pattern theorem in two dimensions

Let’s return to the labeled simplicial complex, with its \(2(n-1)(m-1)\) triangles, obtained from the \(n \times m\) sign matrix. Let \(w^+\) and \(w^-\) denote the number of positive resp. negative well-labeled triangles it contains. Let’s evaluate the sum \(S\) of the formal boundaries of triangles in two ways. By the lemma

\[
S = \sum_{T \in \text{triangles}} \partial(T) = (w^+ - w^-)(\langle a, b \rangle + \langle b, c \rangle + \langle c, a \rangle)
\]
proof of the sign pattern theorem in two dimensions

On the other hand, since the complex is oriented, the interior edges cancel and the sum equals the sum of oriented edges along the external boundaries.

The right-hand edge contributes 0. Apply the one-dimensional case of the sign pattern theorem to the other edges to see that their contribution is $\langle c, a \rangle$ resp. $\langle a, b \rangle$ resp. $\langle b, c \rangle$. 
proof of the sign pattern theorem in two dimensions

\[ S = \sum_{T \in \text{triangles}} \partial(T) = (w^+ - w^-)(\langle a, b \rangle + \langle b, c \rangle + \langle c, a \rangle) \]

\[ S = \langle a, b \rangle + \langle b, c \rangle + \langle c, a \rangle \]

\[ w^+ - w^- = 1 \]

- there’s a triangle with all vertices labeled different
- there’s an odd number of so labeled triangles
- the number of triangles with all vertices labeled different, clockwise, is one more than the number of triangles with all vertices labeled different, counterclockwise.
Recall the one-dimensional situation: given a string of + and − symbols beginning with + and ending with −,

- the string contains $+$ as substring somewhere
- the string contains an odd number of sign changes
- the number of $+$ substrings is one more than the number of $-$ substrings.

The analogy is perfect!
So there is at least one triangle with vertices $a, b, c$. That triangle is one half of (the edge dual of) a $2 \times 2$ submatrix. Recall the definition of labels

\[
\begin{align*}
  a & \{ ++ \\
  b & \{ +-- \\
  c & \{ --+ \\
\end{align*}
\]

...to see that that submatrix contains two sign-pairs that are each other's reverses.
Suppose $g_1, g_2$ are continuous functions $[0, 1]^2 \to \mathbb{R}$ such that for all $0 \leq y \leq 1$, $g_1(0, y) \geq 0$ and $g_1(1, y) \leq 0$ for all $0 \leq x \leq 1$, $g_2(x, 0) \geq 0$ and $g_2(x, 1) \leq 0$.

Then $g_1$ and $g_2$ have a common zero in $[0, 1]^2$.

- follows from the sign pattern theorem by sampling the domain $[0, 1]^2$ more and more densely at a rectangular array, finding a point in a neighborhood of which both $g_1$ and $g_2$ change signs, applying the Bolzano-Weierstrass theorem and continuity to deduce the existence of a simultaneous zero of $g_1$ and $g_2$.
- the vector-valued intermediate value theorem implies the Brouwer fixed point theorem.
- argument generalizes to $n$ dimensions.