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Let B_n be the n-dimensional unit ball $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leqslant 1\}$.
Let $f : B_n \to B_n$ be any continuous function.
Then $f(x) = x$ for some point $x \in B_n$.

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Is it even *believable*? I wouldn't say so!

Brouwer's sensory intuitions

The theorem is supposed to have originated from Brouwer's observation of a cup of coffee. If one stirs to dissolve a lump of sugar, it appears there is always a point without motion. He drew the conclusion that at any moment, there is a point on the surface that is not moving.

Brouwer is said to have added: "I can formulate this splendid result different, I take a horizontal sheet, and another identical one which I crumple, flatten and place on the other. Then a point of the crumpled sheet is in the same place as on the other sheet."

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Me: I've never been *less* convinced in my life.

what we'll do

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- Use approximate fixed point theory and compactness to conclude Brouwer's fixed point theorem in topology.

one-dimensional case is easy

Let $f : [0,1] \to [0,1]$ be continuous. Consider $g(x) = f(x) - x$. $g(0) \geqslant 0$ and $g(1) \leqslant 0$.

By the Intermediate Value Theorem, $g(x_0) = 0$ for some $x_0 \in [0,1]$. Thence $f(x_0) = x_0$.

different model for two-dimensional case

Unit disk B_2 and unit square $[0, 1]^2$ are *homeomorphic* (there's a bijection between them that is continuous, with a continuous inverse).

different model for two-dimensional case

Unit disk B_2 and unit square $[0, 1]^2$ are *homeomorphic* (there's a bijection between them that is continuous, with a continuous inverse).

Challenge Say that a topological space X has the *fixed point property* if any continuous map $f : X \to X$ has a fixed point.

Suppose X and Y are homeomorphic. Then if X has the fixed point property, so does Y.

two dimensions, take 2

We want to prove: any continuous map $\langle f_1(x,y), f_2(x,y)\rangle : [0,1]^2 \to [0,1]^2$ has a fixed point.

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Exhibiting a *total* lack of imagination, we'll do the same as in the one-dimensional case:

Let $g_1(x,y) = f_1(x,y) - x$ and $g_2(x,y) = f_2(x,y) - y$.

Then g_1, g_2 are continuous functions $[0,1]^2 \to \mathbb{R}$.

For all $0 \leqslant y \leqslant 1$, $g_1(0,y) \geqslant 0$ and $g_1(1,y) \leqslant 0$.
For all $0 \leqslant x \leqslant 1$, $g_2(x,0) \geqslant 0$ and $g_2(x,1) \leqslant 0$.

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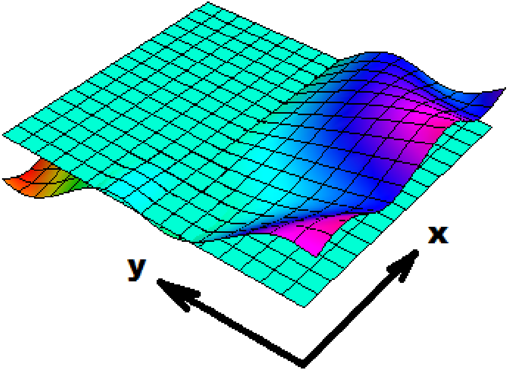
For all $0 \leqslant x \leqslant 1$, $g_2(x,0) \geqslant 0$ and $g_2(x,1) \leqslant 0$.

A point $\langle x,y\rangle$ that is a simultaneous zero of g_1 and g_2 is the same as a fixed point of $\langle f_1(x,y), f_2(x,y)\rangle$.

just intuitively

Suppose $g : [0,1]^2 \to \mathbb{R}$ is continuous such that
$g(x,0) \geqslant 0$ and $g(x,1) \leqslant 0$ for all $0 \leqslant x \leqslant 1$.



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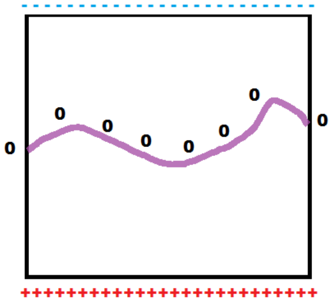
Such a function "ought to" have a "band of zeros" connecting the $x = 0$ and $x = 1$ edges:



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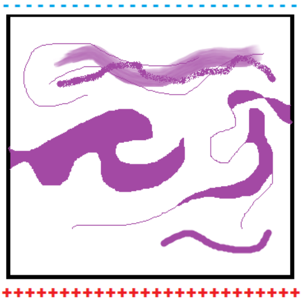


just intuitively

If the zero locus of g did *not* have a component connecting the $x = 0$ and $x = 1$ edges

then there would exist a zero-free path connecting a negative value of g with a positive value, contradicting the 1-dimensional case of the Intermediate Value Theorem!

why Brouwer's theorem is plausible

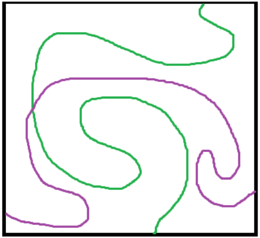
Suppose g_1, g_2 are continuous functions $[0,1]^2 \to \mathbb{R}$ such that
for all $0 \leqslant y \leqslant 1$, $\quad g_1(0,y) \geqslant 0 \;$ and $\; g_1(1,y) \leqslant 0$
for all $0 \leqslant x \leqslant 1$, $\quad g_2(x,0) \geqslant 0 \;$ and $\; g_2(x,1) \leqslant 0$.

Then:

- there is a path in the zero locus of g_1 connecting the $y = 0$ and $y = 1$ edges
- there is a path in the zero locus of g_2 connecting the $x = 0$ and $x = 1$ edges

why Brouwer's theorem is plausible

path in the zero locus of g_1 and path in the zero locus of g_2



- if you connect top to bottom and left to right, those paths "must" intersect
- a common zero of g_1 and g_2 exists
- Brouwer fixed point theorem for $[0, 1]^2$ follows!

just intuitively

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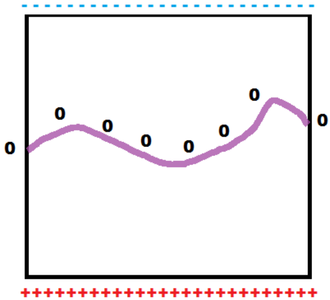

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Then there exists a continuous path $\langle p_1, p_2 \rangle : [0,1] \to Z$ such that $p_1(0) = 0$ and $p_1(1) = 1$.

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Challenge Give a counterexample.

:-(

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- Then there is a connected component Z_0 of Z that intersects both $x = 0$ and $x = 1$. True.
- Then there is a path connected component Z_0 of Z that intersects both $x = 0$ and $x = 1$. False.

HAIRY!

point-set topological difficulties

- cannot always find cross-section of the zero locus that "looks like" an interval
- topology gets yet more complicated for $[0, 1]^n$ with $n > 2$
- proof can be completed this way but needs difficult algebraic topological machinery — more difficult than other (algebraic topological) proofs of Brouwer's fixed point theorem

rescue

Suppose $\mathcal{F}$ is a set of continuous functions from $[0,1]^n$ to itself that is *uniformly dense* among all continuous functions from $[0,1]^n$ to itself:

For any continuous $g : [0,1]^n \to [0,1]^n$ and $\epsilon > 0$ there exists $f \in \mathcal{F}$ such that $\|g(\mathbf{x}) - f(\mathbf{x})\| < \epsilon$ for all $\mathbf{x} \in [0,1]^n$.

Suppose you manage to prove: any $f \in \mathcal{F}$ has a fixed point.

Then it follows that any continuous $[0,1]^n \to [0,1]^n$ has a fixed point.

moral

- find a ‘nice’ class of continuous functions
- should be uniformly dense among all continuous functions but not allow hairy point-set theoretic phenomena
- prove the intermediate value theorem on $[0, 1]^n$ for these functions
- Brouwer’s fixed point theorem follows for all continuous functions by approximate fixed point theory.

nice?

What are ‘nice’ families of functions, dense among all continuous real-valued functions on $[0, 1]^n$?

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- step functions.

5	2	1	3	-9
9	2.23	5	7	17
-1	-2.4	-3	1	5
8	-.5	7	-2	-4
-2	6	-1.1	-1	4

+	+	+	+	−
+	+	+	+	+
−	−	−	+	+
+	−	+	−	−
−	+	−	−	+

sign patterns in matrices of signs

We'll be interested in $n \times m$ matrices, each entry of which contains two symbols:

either $+$ or $-$ (corresponding to the sign of g_1) as well as
either $+$ or $-$ (corresponding to the sign of g_2)

such that

$$\text{boundary conditions} \begin{cases} \text{the first column must contain } + \\ \text{the last column must contain } - \\ \text{the bottom row must contain } + \\ \text{the top row must contain } -. \end{cases}$$

+					−
+					−
+					−
+					−
+					−

+ −	−	−	−	−	− −
+					−
+					−
+					−
+ +	+	+	+	+	− +

$+-$	$+-$	$--$	$+-$	$--$	$--$
$+-$	$-+$	$+-$	$--$	$++$	$--$
$+-$	$++$	$--$	$+-$	$+-$	$--$
$++$	$-+$	$+-$	$-+$	$--$	$-+$
$++$	$-+$	$-+$	$++$	$++$	$-+$

seeking a statement to prove

What are the “simultaneous zeros” of the two functions, signified by the red resp. blue sign entries?

Simultaneous zeros sorta kinda like correspond to “adjacent entries” of the sign matrix where “sign changes occur”.

What does this mean, really?

reality check: dimension 1

Consider a string of $+$ and $-$ signs beginning with $+$ and ending with $-$

$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$-$

Then

reality check: dimension 1

Consider a string of $+$ and $-$ signs beginning with $+$ and ending with $-$

$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$-$

Then

- the string contains $\boxed{+ \mid -}$ as substring somewhere

reality check: dimension 1

Consider a string of $+$ and $-$ signs beginning with $+$ and ending with $-$

$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$-$

Then

- the string contains $\boxed{+\ \middle|\ -}$ as substring somewhere
- the string contains an *odd* number of sign changes

reality check: dimension 1

Consider a string of $+$ and $-$ signs beginning with $+$ and ending with $-$

$$\boxed{+}\boxed{+}\boxed{-}\boxed{+}\boxed{+}\boxed{-}\boxed{+}\boxed{+}\boxed{-}$$

Then

- the string contains $\boxed{+}\boxed{-}$ as substring somewhere
- the string contains an *odd* number of sign changes
- the number of $\boxed{+}\boxed{-}$ substrings is one more than the number of $\boxed{-}\boxed{+}$ substrings.

seeking a statement to prove

Any sign matrix satisfying the boundary conditions, like

$+-$	$+-$	$--$	$+-$	$--$	$--$
$+-$	$-+$	$+-$	$--$	$++$	$--$
$+-$	$++$	$--$	$+-$	$+-$	$--$
$++$	$-+$	$+-$	$-+$	$--$	$-+$
$++$	$-+$	$-+$	$++$	$++$	$-+$

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Any sign matrix satisfying the boundary conditions

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(a) contains a 2×2 submatrix with all four types of entries $\boxed{++}$, $\boxed{+-}$, $\boxed{-+}$ and $\boxed{--}$ (?)

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(b) contains adjacent $\boxed{++}$ and $\boxed{--}$ entries (?)

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(c) contains adjacent $\boxed{+-}$ and $\boxed{-+}$ entries (?)

(d) contains adjacent sign-reversed entries, that is, adjacent $\boxed{++}$ and $\boxed{--}$ *or* adjacent $\boxed{+-}$ and $\boxed{-+}$ (?)

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(e) contains a 2×2 submatrix where all four symbols $+ - + -$ occur (?).

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(e) contains a 2×2 submatrix where all four symbols $+ - + -$ occur YES in all dimensions.

what pattern?

Any sign matrix satisfying the boundary conditions will contain 2×2 submatrix where all four symbols ${\color{red}+}\ {\color{red}-}\ {\color{blue}+}\ {\color{blue}-}$ occur, such as

$$\begin{array}{|c|c|}\hline {\color{red}+}{\color{blue}-} & \\ \hline {\color{red}-}{\color{blue}+} & \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline {\color{red}+}{\color{blue}+} & \\ \hline & {\color{red}-}{\color{blue}-} \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline {\color{red}+} & {\color{blue}+} \\ \hline {\color{red}-}{\color{blue}-} & \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline {\color{blue}-} & {\color{blue}+} \\ \hline {\color{red}-} & {\color{red}+} \\ \hline \end{array}.$$

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- can in fact guarantee that all four symbols will be found in two adjacent entries (possibly "corner adjacent")

what pattern?

Any sign matrix satisfying the boundary conditions will contain 2×2 submatrix where all four symbols ${\color{red}+}\ {\color{red}-}\ {\color{blue}+}\ {\color{blue}-}$ occur, such as

$$\begin{array}{|c|c|}\hline {\color{red}+}{\color{blue}-} & \\ \hline {\color{red}-}{\color{blue}+} & \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline {\color{red}+}{\color{blue}+} & \\ \hline & {\color{red}-}{\color{blue}-} \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline {\color{red}+} & {\color{blue}+} \\ \hline {\color{red}-}{\color{blue}-} & \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline {\color{blue}-} & {\color{blue}+} \\ \hline {\color{red}-} & {\color{red}+} \\ \hline \end{array}.$$

- can in fact guarantee that all four symbols will be found in two adjacent entries (possibly “corner adjacent”)
- don’t know if this holds in dimensions greater than two!

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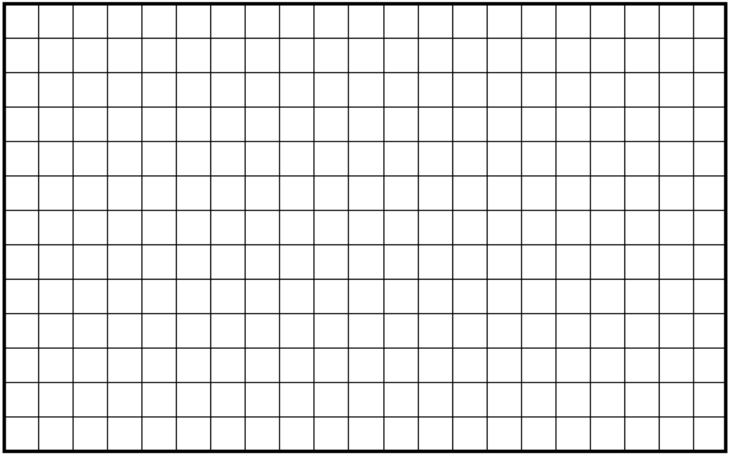
- can in fact guarantee that all four symbols will be found in two adjacent entries (possibly “corner adjacent”)
- don’t know if this holds in dimensions greater than two!
- can prove that in dimension n, all $2n$ symbols will be found in $n+1$ adjacent n-cells.

how to prove it

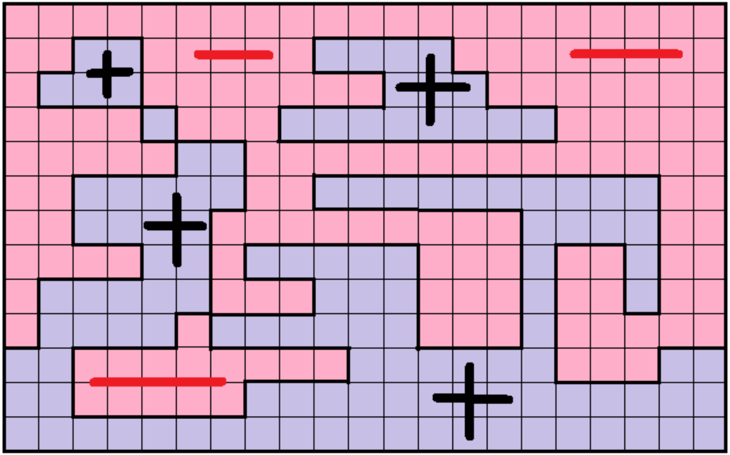
Two approaches to the sign pattern theorem:

- digital: direct, visual, combinatorial, hard
- algebraic: indirect “magic”, non-constructive, powerful

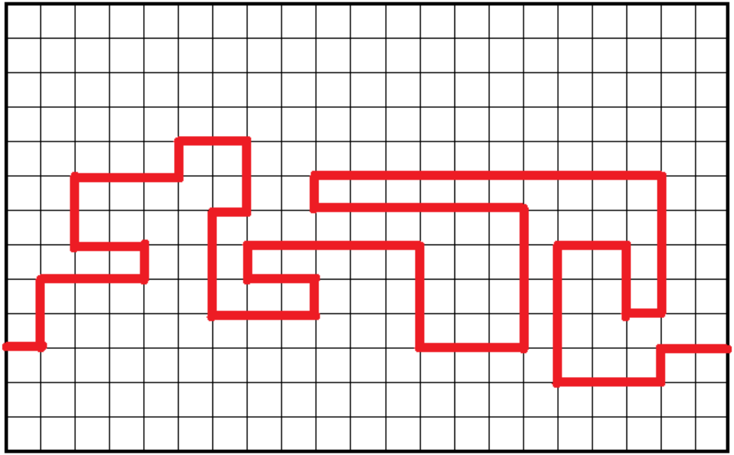
digital topology



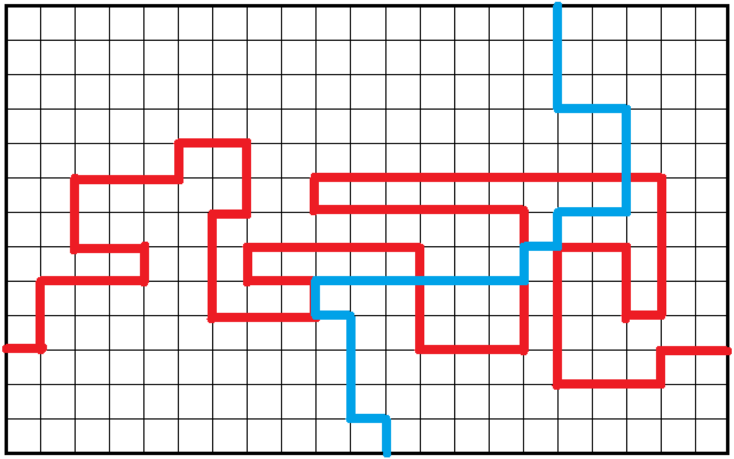
digital topology



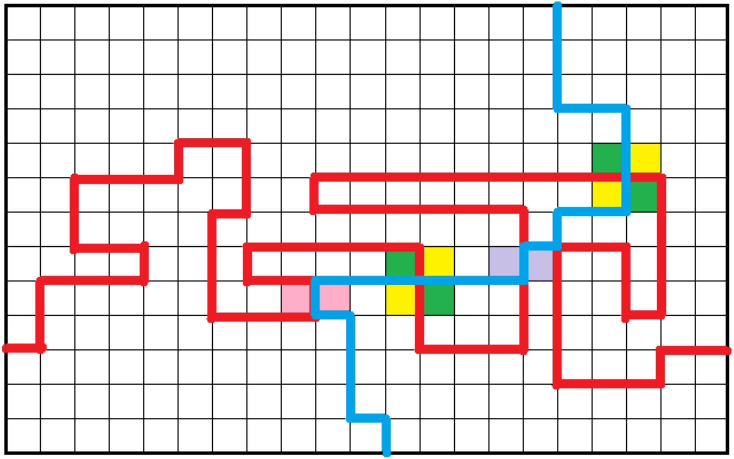
digital divide



digital crossing



adjacent entries containing reversed sign-pairs



Challenge Make the above argument precise.

Challenge++ Make the above argument work in 3 dimensions.

setting up the algebra

assign to each sign-pair one of three labels

$$a \left\{ \quad {\color{red}+}{\color{blue}+} \right. \qquad b \left\{ \quad {\color{red}+}{\color{blue}-} \right. \qquad c \left\{ \begin{array}{l} {\color{red}-}{\color{blue}+} \\ {\color{red}-}{\color{blue}-} \end{array} \right.$$

setting up the algebra

given a sign matrix

$$\begin{array}{ccccc} +- & -- & -- & +- & -- \\ ++ & +- & -- & -+ & -- \\ +- & ++ & +- & ++ & -+ \\ ++ & ++ & -+ & ++ & -+ \end{array}$$

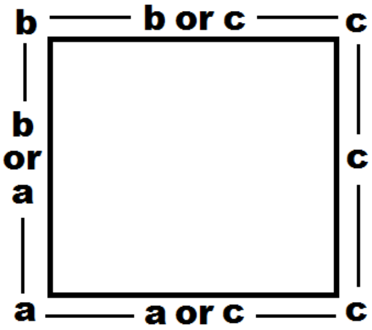
setting up the algebra

replace each sign-pair by its label

b	c	c	b	c
a	b	c	c	c
b	a	b	a	c
a	a	c	a	c

setting up the algebra

note new boundary conditions



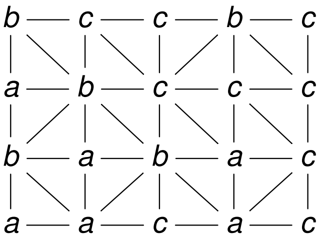
setting up the algebra

consider the *dual* grid, placing the symbols at the vertices and connecting them with edges

$$\begin{array}{ccccccccc}
b & — & c & — & c & — & b & — & c \\
| & & | & & | & & | & & | \\
a & — & b & — & c & — & c & — & c \\
| & & | & & | & & | & & | \\
b & — & a & — & b & — & a & — & c \\
| & & | & & | & & | & & | \\
a & — & a & — & c & — & a & — & c
\end{array}$$

setting up the algebra

triangulate the grid in *any* way, obtaining a simplicial complex with labeled vertices



the labeled boundary operator

Let E (for “edges”) be the abelian group generated by the symbols $\langle x, y\rangle$ where $x, y \in \{a, b, c\}$, subject to the relations

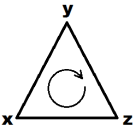
$$\langle x, y\rangle = -\langle y, x\rangle$$

$$\langle x, x\rangle = 0$$

for all $x, y \in \{a, b, c\}$.

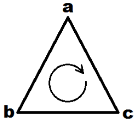
the labeled boundary operator

Note that all triangles in the labeled complex can be oriented compatibly (say, clockwise). Let the boundary operator ∂ be the map from labeled, oriented triangles to E defined by

$\partial\Big($  $\Big) = \langle x, y\rangle + \langle y, z\rangle + \langle z, x\rangle$

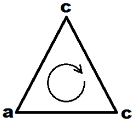
the labeled boundary operator

examples

$\partial\Big($ 

$\Big) = \langle a, c\rangle + \langle c, b\rangle + \langle b, a\rangle$

$$= -\langle a, b\rangle - \langle b, c\rangle - \langle c, a\rangle$$

$\partial\Big($ 

$\Big) = \langle a, c\rangle + \langle c, c\rangle + \langle c, a\rangle$

$$= \langle a, c\rangle + 0 - \langle a, c\rangle = 0$$

labeled boundaries: the key property

A triangle is *well-labeled* if all three labels a, b, c show up on its vertices. It is *positive well-labeled* if a, b, c occur clockwise and *negative well-labeled* if they occur in the other orientation.

Lemma. Let T be a colored triangle.

- $\partial(T) = 0$ unless T is well-labeled
- $\partial(T) = \langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle$ if T is positive well-labeled
- $\partial(T) = -\langle a, b\rangle - \langle b, c\rangle - \langle c, a\rangle$ if T is negative well-labeled.

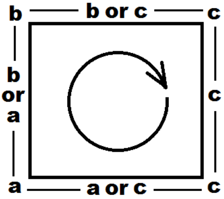
proof of the sign pattern theorem in two dimensions

Let's return to the labeled simplicial complex, with its $2(n-1)(m-1)$ triangles, obtained from the $n \times m$ sign matrix. Let w^+ and w^- denote the number of positive resp. negative well-labeled triangles it contains. Let's evaluate the sum S of the formal boundaries of triangles in two ways. By the lemma

$$S = \sum_{T \in \text{triangles}} \partial(T) = (w^+ - w^-)\big(\langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle \big)$$

proof of the sign pattern theorem in two dimensions

On the other hand, since the complex is oriented, the interior edges cancel and the sum equals the sum of oriented edges along the external boundaries.



The right-hand edge contributes 0. Apply the one-dimensional case of the sign pattern theorem to the other edges to see that their contribution is $\langle c, a\rangle$ resp. $\langle a, b\rangle$ resp. $\langle b, c\rangle$.

proof of the sign pattern theorem in two dimensions

$$S = \sum_{T \in \text{triangles}} \partial(T) = (w^+ - w^-)\big(\langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle\big)$$

$$S = \langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle$$

$$w^+ - w^- = 1$$

- there's a triangle with all vertices labeled different
- there's an *odd* number of so labeled triangles
- the number of triangles with all vertices labeled different, clockwise, is one more than the number of triangles with all vertices labeled different, counterclockwise.

reality check

Recall the one-dimensional situation: given a string of $+$ and $-$ symbols beginning with $+$ and ending with $-$,

- the string contains $\boxed{+ \mid -}$ as substring somewhere
- the string contains an *odd* number of sign changes
- the number of $\boxed{+ \mid -}$ substrings is
  one more than the number of $\boxed{- \mid +}$ substrings.

The analogy is perfect!

end of the proof

So there is at least one triangle with vertices a, b, c. That triangle is one half of (the edge dual of) a 2×2 submatrix. Recall the definition of labels

$$a \left\{ \quad ++ \right. \qquad b \left\{ \quad +- \right. \qquad c \left\{ \begin{array}{l} -+ \\ -- \end{array} \right.$$

to see that that submatrix contains two sign-pairs that are each other's reverses. $\square$

the vector-valued intermediate value theorem

Suppose g_1, g_2 are continuous functions $[0,1]^2 \to \mathbb{R}$ such that
for all $0 \leqslant y \leqslant 1$, $g_1(0,y) \geqslant 0$ and $g_1(1,y) \leqslant 0$
for all $0 \leqslant x \leqslant 1$, $g_2(x,0) \geqslant 0$ and $g_2(x,1) \leqslant 0$.

Then g_1 and g_2 have a common zero in $[0,1]^2$.

- follows from the sign pattern theorem by sampling the domain $[0,1]^2$ more and more densely at a rectangular array, finding a point in a neighborhood of which both g_1 and g_2 change signs, applying the Bolzano-Weierstrass theorem and continuity to deduce the existence of a simultaneous zero of g_1 and g_2
- the vector-valued intermediate value theorem implies the Brouwer fixed point theorem
- argument generalizes to n dimensions.